A STUDY OF CERTAIN SUBCLASSES OF UNIVALENT AND MULTIVALENT ANALYTIC FUNCTIONS

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Dedicated to

my

Beloved parents

Dr. Harish Chandra Gupta

and

Smt. Urmila Gupta

C E R T I F I C A T E

This is to certify that the work embodied in the thesis 'A STUDY OF CERTAIN SUBCLASSES OF UNIVALENT AND MULTIVALENT ANALYTIC FUNCTIONS' by Rita Gupta has been done under my supervision and has not been submitted elsewhere for a degree or diploma.

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CONTENTS

SYNOPSIS

The univalent functions have been extensively studied during the last seventy years by different research workers. Bieberhach conjectured the famous Bieberbach conjecture in 1916. Recently this conjecture has been settled affirmatively by Louis de Branges [U.S.S.R. Academy of Sciences Steklov Mathematical Institute Leningard]. An independent proof of this conjecture is given by Pommerenke and Fitzgerad also.

Since 1916 till the mid of 1984 (i.e. for 68 years) many prominent mathematicians tried to settle this conjecture in its full generality but failed to do so. This led to the investigation of several subclasses of univalent functions, in turn benefitted the theory abundantly due to the introduction of new concepts and techniques. Usually the types of problems studied consists (a) coefficient estimates (b) bounds of $\arg \frac{f(z)}{z}$ and $\arg f'(z)$ (c) Destortion theorems i.e. the determination of lower and upper estimates of $|f(z)|$, $|f'(z)|$ and $\left|\frac{zf'(z)}{f(z)}\right|$ etc. (d) Radii of univalency, starlikeness, convexity and close-to-convexity etc. (e) study of certain operators which are either class preserving or give a relation between two classes (f) Extremal problems.

The present thesis consists of six chapters. Chapter I gives a brief introduction to certain aspects of the theory of

univalent and multivalent functions needed in the prepration of the work presented in the following chapters.

In Chapter II the subclasses $S_s^{*,n}(A,B)$, $(-1 \leq B < A \leq 1)$ of close-to-convex functions have been introduced, certain integral transforms in this class have been considered. A necessary and sufficient condition in terms of convolutions for a function to be in $S_s^{*,n}(A,B)$ has been obtained. It is further shown that $S_s^{*,n}(A,B)$ is closed under convolution with the class of convex functions. Coefficient estimates for $S_s^{*,n}(A,B)$ and $S_s^{*,n}(1-2\alpha,-1)$ (with fixed (n+1)th coefficient for f in the class $S_s^{*,n}(1-2\alpha,-1)$ have been obtained. For any complex number μ, maximum value of the functional $|a_3-\mu a_2^2|$ for functions in this class has been determined. For different choices of parameters n, A and B, the results of this chapter yield along with some new results, the results of Goel and Mehrok (Tamkang J. Math. 13 (1982)), Silverman, Silvia and Telage (Math. z. 162 (1978),2), Silverman and Silvia (Rocky Mountain J. Math. 10 (1980)) etc.

In Chapter III, the class $M(\lambda_1,\lambda_2)$ has been introduced. Using the technique of differential subordination, recently developed by S.S. Miller and P.T. Mocanu (Michigan Math. J. 28 (1980)), it has been shown that for $\lambda_2 \leq 0$, functions in $M(\lambda_1,\lambda_2)$ are starlike and for $\lambda_1+\lambda_2 \geq 1$, functions in $M(\lambda_1,\lambda_2)$ are convex. $M(\lambda_1,0)$ is nothing but the Mocanu class of λ-convex functions. An open problem that for $\lambda_2 > 0$ and

$\lambda_1+\lambda_2 < 1$, the functions in $M(\lambda_1,\lambda_2)$ are starlike has been posed. Using the same technique, the subordination between certain class of functions have been obtained. The results of this chapter yield along with some new results, the results obtained by Miller and Mocanu (Michigan Math. J. 28 (1980)), Mocanu and Reade (Mathematica (cluj) 20 (43), 1(1978)).

In Chapter IV, the order of starlikeness of a certain integral operator on certain subclasses of univalent and multivalent starlike functions has been obtained by using the sharp subordination results recently established by Miller and Mocanu (Lecture Notes in Math., 1013 Springer, Berlin New York 1983, 292-310) as well as a lemma due to D.R. Wilken and J. Feng (J. London Math. Soc. (2) 21 (1981). The results of this chapter generalize the results obtained by several authors, S.S. Miller, Mocanu and Reade (Mathematica (cluj) 20 (43), 1(1978)), Mocanu, Reade and Ripeanu (Mathematica (cluj) 19 (42), 1(1977)), St. Ruschwey and V. Singh (Rev. Roum. Math. Pures Appl. 24 (1979)), K.S. Padmanabhan and G.L. Reddy (Bull. Aust. Math. Soc. 25 (1982)) etc.

Chapter V deals with certain subclasses $S_N^*(A,B)$, $Q_N(\alpha,A,B)$, $R_N^\alpha(A,B)$, $(-1 \leq B < A \leq 1)$, $(|\alpha| < \pi/2)$ of univalent functions having N initial missing coefficients. Coefficient estimates, destortion properties and a coefficient inequality have been established for $f(z)$ in the classes $S_N^*(A,B)$ and $R_N^\alpha(A,B)$. The sharp estimates for arc length and area for

$f(z) \in R_N^{\alpha}(A,B)$ have been obtained. A condition depending upon λ, A, B and α has been deduced such that $(1+\lambda)z + \lambda(f(z) * zh'(z))$ belongs to $R_N^{\alpha}(A,B)$ whenever $f(z)$ and $h(z)$ belong to $R_N^{\alpha}(A,B)$. The integral representation, distortions and some coefficient estimates have been obtained for $f(z)$ belonging to $Q_N(\alpha,A,B)$. It has been observed that the extremal functions for coefficient estimates and distortion properties for the above classes are N-fold symmetric functions. The results of this chapter yield along with some other results, the results obtained by Goel and Mehrok (Indian J. Pure Appl. Math. 12 (1981)). Jakubowski, Z.J., Kaminiski, J. (Rev. Roum. Math. Pures Appl. 23 (1978)), Juneja and Mogra (Bull. Sci. Math. (2) 103 (1979)).

VI, the last chapter is devoted to the study of a certain integral operator on various subclasses of functions of the form $f(z) = \frac{1}{z} + a_0 + \sum_{n=1}^{\infty} a_n z^n$, regular in $0 < |z| < 1$. Some results of Goel and Sohi (Glasnik Matematicki 17 (1981)) have been corrected and generalized.

CHAPTER I

INTRODUCTION

1.1 Let D be a domain (an open connected subset of the complex plane). A univalent function in D is characterized by the fact that it takes in D no value more than once and consequently, it maps D onto a domain which is not self-overlapping and has no branch point. A necessary condition for an analytic function $f(z)$ to be univalent in D is that $f'(z) \neq 0$ in D, but this condition is not sufficient to ensure that $f(z)$ is univalent in D. This can be seen by considering the function $f(z) = e^z$ whose derivative never vanishes but it is not univalent in $D = \{|z| < R\}$ where $R > \pi$.

The interplay of geometry and analysis is perhaps the most fascinating aspect of complex function theory and the theory of univalent functions brings out this interplay very elegantly. Riemann mapping theorem asserts that any simply connected domain in the complex plane with more than one boundary point can be mapped conformally onto the interior of the unit disc. One may therefore confine, without loss of generality, to functions regular and univalent in the unit disc $U = \{z: |z| < 1\}$. If $f(z)$ is regular and univalent in U, so is the function $f(z) = \frac{f(z)-f(0)}{f'(0)}$ since $f'(0) \neq 0$. Thus it is enough to consider the functions in U which are

normalized by the condition $f(0) = f'(0)-1 = 0$. Let H denote the class of functions $f(z)$ regular in U normalized by the above conditions and S denote the subclass of H consisting of univalent functions. Let $f \in S$, the Taylor expansion of f about the origin is given by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1.1}$$

Origin of the theory of univalent functions can be traced back to the paper of P. Koebe [39] in which he proved in particular that there exists a constant K (called Koebe's constant) such that the boundary of map of U by any univalent function $w = f(z)$ is in S is always at a distance not less than K from $w = 0$. Koebe's work attracted the attention of other great analysists such as Bieberbach, Gronwall, Spencer, Goluzin, Rogosinski, Robinson, Littlewood, Löwner, Schiffer and others. Gronwall [23] obtained the well known area principle. With the help of that Bieberbach [9] obtained the precise value of $K = \frac{1}{4}$. Also, he proved that $|a_2| \leq 2$ for $f(z) \in S$. He showed that the equality in this estimate is attained for the functions

$$f(z) = \frac{z}{(1+\epsilon z)^2}, \quad |\epsilon| = 1 \tag{1.1.2}$$

known as the Koebe functions. Motivated by the extremal property of the Koebe functions, Bieberbach [9] conjectured that for every $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ belonging to S

$$|a_n| \leq n \quad . \tag{1.1.3}$$

Since 1916 upto the mid of 1984 (i.e. for 68 years) many eminent mathematicians such as Lowner [47], Garabedian and Schiffer [15], Pederson [68], Littlewood [45], Bazilevic [5], Hayman [24], Biberbach [9], Horowitz [29] tried to prove this conjecture in full generality but failed to do so. Recently this conjecture has been proved by Louis de Branges [46].

Other conjectures concerning univalent functions are Milin conjecture [52], Robertson conjecture [75], Sheil-Small conjecture [86], Rogosinski conjecture [76], Asymptotic Bieberbach conjecture [26] Littlewood conjecture [45].

It is interesting to note that seven of the coefficient conjectures are related by a chain of implications given by, Milin conjecture ==> Robertson conjecture ==> Sheil-Small conjecture ==> Rogosinski conjecture ==> Bieberbach conjecture ==> Asymptotic Bieberbach conjecture ==> Littlewood conjecture.

The concept of univalency can be extended to p-valency for functions regular in the unit disc U. A regular p-valent function $f(z) = w$ has at most p solutions and there exists a w_0 belonging to $f(U)$ such that $f(z) = w_0$ has exactly p solutions in U.

Various developments in the theory of univalent and multivalent functions have been surveyed by Montel [64] Schaeffer and Spencer [85], Bernardi [6], Hayman [27], Krzyz [40], Goodman [21], Pommerenke [69] etc. Important aspects of the subject have been covered in the books of Nehari [65], Goluzin [20], Pommerenke [70], Hayman [25], Jenkins [34] and in recent two volume monograph of Goodman [22]. A recent exhaustive biblography of schilcht functions [8] by Bernardi is also available listing the literature on the subject till 1981.

1.2 In this section we list some well known subclasses of S which play a significant role in the study of univalent functions.

A function $f(z)$ in S is said to be starlike with respect to the origin, if it maps U onto a domain starlike with respect to the origin. A domain D is called starlike with respect to the origin if the line segment joining origin to any point of D is in D. The class of such functions is denoted by S^*. It is well known [65] that a function $f(z)$ belonging to H is in S^* if and only if

$$(1.2.1) \qquad \mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0, \text{ for } z \in U.$$

The geometric interpretation of (1.2.1) is that for each fixed r $(0 < r < 1)$, $\arg f(re^{i\theta})$ strictly increases with θ, $0 \leq \theta < 2\pi$. Robertson [74] introduced the concept of order

for starlike functions as follows: A function $f(z)$ in H is said to be starlike of order $\alpha(0 \leq \alpha < 1)$ in U if

$$\text{(1.2.2)} \qquad \mathrm{Re}\,\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha, \text{ for } z \in U.$$

The class of such functions is denoted by $S^*(\alpha)$.

A function $f(z)$ in S is said to be convex in U, if it maps U onto a convex domain. A domain D is convex if whenever z_1 and z_2 belong to D, then so is the line segment joining z_1 and z_2. We shall denote the class of convex functions by K.

A necessary and sufficient condition for a function $f(z)$ in S to belong to K is

$$\text{(1.2.3)} \qquad \mathrm{Re}\,\left\{1+\frac{zf''(z)}{f'(z)}\right\} > 0, \quad z \in U.$$

Geometrically the condition (1.2.3) means that $w = f(re^{i\theta})$ maps each circle $|z| = r < 1$ onto a simple closed contour whose tangent rotates monotonically as θ increases in the counter-clockwise direction.

A function $f(z)$ in S is said to belong to $K(\alpha)$ $(0 \leq \alpha < 1)$, the class of functions convex of order α if

$$\text{(1.2.4)} \qquad \mathrm{Re}\,\left\{1+\frac{zf''(z)}{f'(z)}\right\} > \alpha, \ z \in U.$$

W. Kaplan [37] defined the class of close-to-convex in the following way:

A function $f(z)$ in H is said to belong to C, the class of functions close-to convex in U, if there exists a $g(z) \in K$ such that

$$\text{(1.2.5)} \qquad \operatorname{Re}\left\{\frac{f'(z)}{e^{i\beta}g'(z)}\right\} > 0$$

for some real β with $|\beta| < \pi/2$.

He further characterized close-to-convex functions in the following manner:

Kaplan [37] A function $f(z)$ in H is close-to-convex in U if and only if

$$\text{(1.2.6)} \qquad \int_{\theta_1}^{\theta_2} \operatorname{Re}\left[1+\frac{zf''(z)}{f'(z)}\right] d\theta > -\pi$$

where $\theta_1 < \theta_2$, $z = re^{i\theta}$ and $r < 1$.

Geometrically the condition (1.2.6) implies that the image of $C_r = \{z : |z|=r\}$ under $f(z)$ belonging to C has no ''large hairpin'' turns i.e. there are no sections of the curve $f(C_r)$ in which the tangent vector turns backward through an angle greater than π.

Libera [43] introduced the concept of order $\alpha (0 \leq \alpha < 1)$ and type τ for close-to-convex functions.

A function $f(z)$ in H is said to be close-to-convex of order α $(0 \leq \alpha < 1)$ and type $\tau (0 \leq \tau < 1)$ if there exists a $g(z)$ in $S^*(\tau)$ such that

$$(1.2.7) \qquad \mathrm{Re}\left[\frac{zf'(z)}{g(z)}\right] > \alpha, \; z \in U.$$

The class of such functions is denoted by $C(\alpha,\tau)$. Obviously $C(0,0) = C$.

Another interesting subclass of univalent functions is the class of α-convex functions introduced by Mocanu [60]

A function $f(z)$ in H is said to α-convex in U if $\frac{f(z).f'(z)}{z} \neq 0$ and

$$(1.2.8) \qquad \mathrm{Re}\,\{(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha(1+\frac{zf''(z)}{f'(z)})\} > 0, \text{ for } z \in U.$$

The class of such functions is denoted by $M(\alpha)$. Miller, Mocanu and Reade have shown that if $f(z)$ belong to $M(\alpha)$ for $\alpha \geq 1$, then $f(z)$ is convex while if $\alpha < 1$, then $f(z)$ is starlike. They [58] obtained the integral representation for a function $f(z)$ in $M(\alpha)$ for $\alpha > 0$. In fact they have shown:

$f(z)$ is in $M(\alpha)$, $(\alpha > 0)$ if and only if there exists a function $F(z)$ in S^* such that

$$(1.2.9) \qquad f(z) = \left[\frac{1}{\alpha}\int_0^z \frac{(F(\xi))^{1/\alpha}}{\xi}\right]^{\alpha}$$

where branches are selected properly. Thus there exists a one-one correspondence between S^* and $M(\alpha)$, for $\alpha > 0$. Bajpai and Silvia [3] introduced the concept of order β for α-convex functions as follows:

A function $f(z)$ in H is said to be α-convex of order β if $\frac{f(z).f'(z)}{z} \neq 0$ and

(1.2.10) $\quad \operatorname{Re} \{(1-\alpha) \frac{zf'(z)}{f(z)} + \alpha (1+ \frac{zf''(z)}{f'(z)})\} > \beta$, for $z \in U$

where $-\infty < \alpha < \infty$ and $0 \leq \beta < 1$. The class of such functions is denoted by $M_\beta(\alpha)$. Obviously $M_0(\alpha) = M(\alpha)$.

A function $f(z)$ in H is said to belong to R if

(1.2.11) $\quad \operatorname{Re} f'(z) > 0$, for $z \in U$.

The class R is introduced by Macgreger.

Let $R(\alpha)$ $(0 \leq \alpha < 1)$ denote the class of functions $f(z)$ in H such that

(1.2.12) $\quad \operatorname{Re} f'(z) > \alpha$, for $z \in U$.

Obviously $R(\alpha) \subseteq R$ and R consists of univalent functions only.

1.3 Let B_0 denote the class of functions $\omega(z)$ analytic in U with $\omega(0) = 0$ and $|\omega(z)| < 1$. B_0 is known as the class of schwarz functions.

Important Lemmas for $\omega(z) \in B_0$

Lemma 1.3.1. Schwarz Lemma [65] If $\omega(z)$ is in B_0, then $|\omega(z)| \leq |z|$, equality holds if and only if $\omega(z) = kz$ where $|k| = 1$.

Lemma 1.3.2. Koegh and Merks [38] If $\omega(z) = \sum_{n=1}^{\infty} c_n z^n$ is in B_o and μ is any complex number, then

(1.3.1) $$|c_2 - \mu c_1^2| \leq \max\{1, |\mu|\}$$

Let $f(z)$ and $F(z)$ be analytic in U. We say $f(z)$ is subordinate to $F(z)$, denoted by $f(z) << F(z)$, if there exists a schwarz function $\omega(z)$ such that

(1.3.2) $$f(z) = F(\omega(z))$$

If $f(z) << F(z)$, then the following are satisfied

(i) $f(U) \subset F(U)$

(ii) $f(U_r) \subset F(U_r)$ where $U_r = \{z: |z| < r < 1\}$

(iii) $\max\limits_{|z| \leq r} |f(z)| \leq \max\limits_{|z| \leq r} |F(z)|$

(iv) $\max\limits_{|z| < r} (1-|z|^2)|f'(z)| \leq \max (1-|z|^2)|F'(z)|$ $(0 \leq r < 1)$

The most important case is when the subordinating function is univalent.

Lemma 1.3.3. If $f(z)$ and $F(z)$ be analytic in U and $F(z)$ is univalent in U, then $f(z) << F(z)$ if and only if $f(0) = F(0)$ and $f(U) \subseteq F(U)$.

If we combine (ii) with Lemma 1.3.3 we obtain the useful principle of subordination. If $F(z)$ is univalent in U then $f(0) = F(0)$ and $f(U) \subset F(U)$ implies $f(U_r) \subset F(U_r)$ where $U_r = \{z: |z| < r < 1\}$, $0 < r < 1$.

The definition of N-subordination is due to Singh and Tygel [92] .

Let $f(z)$ and $F(z)$ be analytic in U. Then $f(z)$ will be called N-subordinate to $F(z)$ in U, denoted by $f(z) \underset{N}{\ll} F(z)$ if and only if there exists a schwarz function $\varphi(z)$ such that

(i) $$|\varphi(z)| \leq |z^N|$$

(ii) $$f(z) = F(\omega(z))$$

$\varphi(z)$ is called N-schwarz function. For $N=1$, it is usual subordination.

Another important subclass of analytic functions is $P(A,B)$. Janowski [33] defined $P(A,B)$ which contain many subclasses of interest as special cases.

A function $p(z) = 1+p_1z + p_2z^2+\ldots$ belong to $P(A,B)$ $(-1 \leq B < A \leq 1)$ if and only if

(1.3.3) $$p(z) \ll \frac{1+Az}{1+Bz}$$

Using the definition of subordination we have $p(z) \in P(A,B)$ if and only if there exists a schwarz function $\omega(z)$ such that

$$p(z) = \frac{1+A\,\omega(z)}{1+B\,\omega(z)} .$$

Geometrically $p(z)$ is in $P(A,B)$ if and only if $p(0)=1$ and

(1.3.4) $$\left|p(z)- \frac{1-AB}{1-B^2}\right| < \frac{A-B}{1-B^2} , \text{ for } z \in U \text{ and } B \neq -1.$$

(1.3.5) $\quad \operatorname{Re} p(z) > \frac{1-A}{2}$, for $z \in U$ and $B = -1$.

Let $P = P(0)$ denote the class of functions $p(z)=1+p_1z+p_2z^2+\ldots$ such that

(1.3.6) $\quad \operatorname{Re} p(z) > 0$, for $z \in U$.

One can easily see that $P = P(0) = P(1,-1)$
Mccarty [48] defined the class $P(\alpha)$ as follows:
A function $p(z) = 1+p_1z + p_2z^2+\ldots$ belongs to $P(\alpha)$ if

$$\operatorname{Re} p(z) > \alpha, \quad \text{for } z \in U.$$

Also, $P(\alpha) = P(1-2\alpha,-1)$.

In fact, special choices of parameters A,B lead to familiar subclasses of P such as

(i) $p(z) \in P(1,-1+\frac{1}{M})$, $M > \frac{1}{2}$ if and only if $|p(z)-M| < M$. This class has been defined by Janowski [32].

(ii) $p(z) \in P(1,1-2\alpha)$, $0 \leq \alpha < 1$, if and only if $|p(z) - \frac{1}{2\alpha}| < \frac{1}{2\alpha}$ considered by Shaffer [84].

1.4. In recent years various subclasses of univalent functions related to $P(A,B)$ have been studied by various authors.

A function $f(z)$ in S^* is said to belong to $S^*(A,B)$ $(-1 \leq B < A \leq 1)$ if and only if

$$\frac{zf'(z)}{f(z)} \ll \frac{1+Az}{1+Bz}, \quad \text{for } z \in U.$$

Equivalently, a function $f(z)$ belongs to $S^*(A,B)$ if and only if $\frac{zf'(z)}{f(z)}$ is in $P(A,B)$. $S^*(A,B)$ have been considered by Janowski [33] Goel and Mehrok [16]. Special selections of A and B give different subclasses of S^* studied by various authors.

(i) $S^*(1,0)$ is considered by Ram Singh [94].

(ii) $S^*(\alpha,0)$ is considered by Eenigenburg [13].

(iii) $S^*(1,1-\frac{1}{M})$, $M > \frac{1}{2}$ is considered by Janowski [32]

(iv) $S^*(\alpha,=\alpha)$, $0 < \alpha < 1$, is considered by Padmanabhan [66]

Analogous to the class $S^*(A,B)$, we can define the class $K(A,B)$.

A function $f(z)$ in S is said to belong to $K(A,B)$ if and only if $zf'(z)$ is in $S^*(A,B)$. Clearly $K(A,B)$ is a subclass of K for $(-1 \leq B < A \leq 1)$ and $K(1-2\alpha,-1) = K(\alpha)$ $(0 \leq \alpha < 1)$. For suitable values of A and B, we obtain different subclasses of K.

A function $f(z)$ in H is said to belong to $R(A,B)$ $(-1 \leq B < A \leq 1)$ if and only if

$$f'(z) \ll \frac{1+Az}{1+Bz}$$

Since $\operatorname{Re} f'(z) > 0$, $R(A,B)$ consists of only univalent functions.

$R(A,B)$ is studied by Juneja and Mogra [35]. Special selections of parameters A,B give different subclasses of R.

1.5 A function $f(z)$ in H is said to be starlike w.r.t. symmetric points if and only if, for every $r < 1$ and every z_0 on $|z| = r$, the angular velocity of $f(z)$ about $f(-z_0)$ is positive at $z = z_0$ as z traverses the circle $|z|=r$ is the positive direction

$$\operatorname{Re} \frac{zf'(z)}{f(z)-f(-z)} > 0 \quad , \text{ for } z \in U$$

The class of such functions is denoted by S_s^*. The concept of starlikeness with respect to symmetric points is due to Sakaguchi [83] .

The concept of starlike functions of order $\alpha(0 \leq \alpha < 1)$ with respect to n symmetric points (n is a positive integer) is due to Rattan Chand [11]

A function $f(z)$ in H is said to be starlike of order $\alpha(0 \leq \alpha < 1)$ with respect to n symmetric points if

$$\operatorname{Re} \left\{\frac{zf'(z)}{f_n(z)}\right\} > \alpha , \quad \text{for } z \in U.$$

where $f_n(z) = \frac{1}{n} \sum_{i=0}^{n-1} \omega^{-i} f(\omega^i z)$ and $\omega = e^{\frac{2\pi i}{n}}$ i.e. the nth root of unity. This class is denoted by $S_s^{*,n}(\alpha)$. Recently this class has been extensively studied by Rattan Chand [11] , Prithvipal Singh [93] and T.R. Reddy [73] .

1.6 Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be analytic in U. The Hadamard product of $f(z)$ and $g(z)$, denoted by $f*g$, is defined by

$f*g(z) = \sum_{n=0}^{\infty} a_n b_n z^n$ and represents an analytic function in U.

Polya and Schoenberg conjectured the famous Polya-Schonberg conjecture in 1958.

Polya Schoenberg [71] . If f and g are in K, then so is the f*g. Suffridge in 1966 shown that under the hypothesis of the conjecture, f*g is close-to-convex. Ruscheweyh and Sheil-Small settled in 1973 the Polya schoenberg conjecture affirmatively. In fact they proved the following four theorems[81] .

Theorem 1.6.1. If f and g are in K, then f*g. is also in K.

Theorem 1.6.2. If f is in K and g is in C, then f*g is in C.

Theorem 1.6.3. If f and g are odd starlike functions, then f*g is also an odd starlike function.

Theorem 1.6.4. Suppose f and g are in K and φ is subordinate to g, then $f*\varphi$ is subordinate to f*g.

Let
$$\frac{z}{(1-z)^{2(1-\alpha)}} = \sum_{k=1}^{\infty} \gamma(\alpha,k)z^k .$$

Suffridge in 1976 [97] proved the following remarkable result . If $\alpha < 1$ and $f(z) = \sum_{k=1}^{\infty} \gamma(\alpha,k) a_k z^k$ and $g(z) = \sum_{k=1}^{\infty} \gamma(\alpha,k)b_k z^k$ are both in $ST(\alpha)$, then

$$f(z) \underset{\alpha}{*} g(z) = \sum_{k=1}^{\infty} \gamma(\alpha,k)a_k b_k z^k$$

is also in $ST(\alpha)$.

For $\alpha=0$, this gives the truth of Polya-Schonberg conjecture.

A great deal of work, using convolution techniques, has been done by various authors such as Ruscheweyh [77, 78, 79] Shiel-Small [86, 87, 88], Suffridge [97], Al-Amiri [1, 2], Silverman and Silvia [96], Silverman [89] etc.

1.7 Let $S(p)$ denote the class of functions

$$(1.7.1) \qquad f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \qquad (p \text{ is a positive integer})$$

regular in U.

A function $f(z)$ in $S(p)$ is said to belong to $S_p^*(\alpha)$ $(0 \leq \alpha < 1)$ if

$$(1.7.2) \qquad \frac{1}{p} \operatorname{Re}\left\{\frac{zf'}{f}\right\} > \alpha$$

$S_p^*(\alpha)$ denote the class of functions p-valent starlike of order α with the zero of order p at the origin.

We shall denote by $K_p(\alpha)$ $(0 \leq \alpha < 1)$ the class of functions $f(z)$ in $S(p)$ such that

$$(1.7.3) \qquad \frac{1}{p} \operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha .$$

$K_p(\alpha)$ is known as the class of p-valent convex functions with the zero of order p at the origin.

1.8. The concept of univalency can be extended to functions which are regular in the punctured disc $0 < |z| < 1$. Let Σ denote the class of functions $f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n$ which are

regular and univalent in $0 < |z| < 1$ except at the point $z=0$ where it has a simple pole. Let Σ_0 denote the subclass of functions $f(z)$ in Σ with $a_0=0$. Analogous to the subclasses $S^*(\alpha)$, $K(\alpha)$ of S, subclasses $\Sigma^*(\alpha)$ and $K(\alpha)$ of Σ consisting of functions meromorphically starlike and convex of order α respectively have been defined in the following way.

A function $f(z)$ in Σ is said to be meromorphically starlike of order α $(0 \leq \alpha < 1)$ if the complement of $f(U)$ is starlike of order α with respect to the origin.

A function $f(z)$ in Σ is said to belong to $\Sigma^*(\alpha)$ $(0\leq\alpha<1)$ if and only if

(1.8.1) $$-\mathrm{Re}\,\frac{zf'(z)}{f(z)} > \alpha, \text{ for } z \in U.$$

A function $f(z)$ in Σ is said to belong to $\Sigma_K(\alpha)$ $(0\leq\alpha<1)$ if and only if

(1.9.2) $$-\mathrm{Re}\,\{1+\frac{zf''(z)}{f'(z)}\} > \alpha\ , \text{ for } z \in U.$$

It can be easily seen that $f \in \Sigma_K(\alpha)$ if and only if $zf' \in \Sigma^*(\alpha)$.

The class of functions meromorphically starlike and convex are identified by $\Sigma^*(0) = \Sigma^*$ and $\Sigma_K(0) = \Sigma_K$.

The class of meromorphic starlike functions has been extensively studied by Pommerenke[69] , Kaczmarski [36] , J.E. Miller [49, 50, 51] Clunie [12] and others.

1.9 In recent years various authors studied the integeral operator of the form

$$(1.9.1) \qquad I_{\beta,\gamma}(f) = \left(\frac{\beta+\gamma}{z^{\gamma}} \int_0^z f^{\beta}(w)\, w^{\gamma-1}\, dw\right)^{1/\beta}$$

Libera studied the integral operator $I_{\beta,\gamma}(f)$. He has shown that whenever $f(z)$ is in S^*, K or C, then $I_{1,1}$ remains in the respective class. Recently V. Singh and Campbell[10] proved that if $f(z)$ is univalent, then $I_{1,1}(f)$ may be infinite-valent. Hence, the univalency is not preserved under the integral operator $I_{1,1}(f)$. Bernardi [7] has shown that if $f(z)$ is in S^*, K, C or R, then $I_{1,k}(f)$ (k is a positive integer) remains in the respective class.

The method used by Libera [44] and Bernardi [7] in obtaining above results is not applicable if β and γ are not integers or if β,γ are complex numbers. Lewandowski et. al. [42] in 1976 partially solved this problem by putting (1.9.1) in the form of a differential equation.

Let $I_{\beta,\gamma}(f) = F$, then

$$(1.9.2) \qquad p(z) + \frac{zp'(z)}{\beta p(z)+\gamma} = g(z)$$

where $p(z) = \frac{zF'(z)}{F(z)}$ and $g(z) = \frac{zf'(z)}{f(z)}$.

Since $p(0) = q(0)$. The differential equation of the form (1.9.2) is of Briot-Bouquet type. Several applications of these equations have been recently appeared [56,57] in the theory of univalent and multivalent functions.

1.10. Let $p(z)$ be regular in the unit disc U and $\psi(r,s,t)$ be a complex valued function defined in a domain of C^3. With some simple conditions on ψ Miller and Mocanu [55] determined the class of functions Ψ for which

(1.10.1) $|\psi(p(z), zp'(z), z^2p''(z))| < 1$, for $z \in U \Longrightarrow |p(z)| <$
for $z \in U$.

Also, they determined a different class of functions for which

(1.10.2) $\mathrm{Re}\,\psi(p(z), zp'(z), z^2p''(z)) > 0$, for $z \in U \Longrightarrow \mathrm{Re}\, p(z)$
for $z \in U$.

They generalized their results in [56] i.e. determined the class of functions for which

$$\{\psi(p(z), zp'(z), z^2p''(z)) : z \in U\} \subset \Omega \Longrightarrow \{p(z) : z \in U\} \subset \Delta \text{ ,}$$

where Ω is a domain and Δ is a simply connected domain whose boundary consists of simple closed regular curve C or the pairwise disjoint simple curves each of which converges to ∞ in both the directions.

Let $\psi : C^3 \to C$ be holomorphic in a domain D and let $h(z)$ be univalent in U. Suppose $p(z)$ is regular in U, $(p(z), zp'(z), z^2p''(z)) \in D$ when $z \in U$ and $p(z)$ satisfies the differential subordination

(1.10.3) $\psi(p(z), zp'(z), z^2p''(z)) \prec\prec h(z)$.

A univalent function $g(z)$ is said to be a dominant of the differential subordination (1.10.3) if $p(z) \ll q(z)$ for all $p(z)$ satisfying (1.10.3). If $\tilde{q}(z)$ is a dominant of (1.10.3) and $\tilde{q}(z) \ll q(z)$ for all dominants $q(z)$ of (1.10.3), then $\tilde{q}$ is said to be the best dominant (1.10.3).

If there are two best dominants q_1 and q_2, then $q_1(z) \ll q_2(z)$ and $q_2(z) \ll q_1(z)$ then $q_1(z) = q_2(e^{i\theta}z)$. Hence, the best dominant of (1.10.3) if there exists, will be unique upto the rotation.

1.11. Till now we have described only those aspects of the theory of univalent and multivalent functions in the direction of which tried to pursue it further. In the present work, an attempt has been to have a detailed study of various subclasses of univalent functions by employing different techniques.

In Chapter II the subclasses $S_s^{*,n}(A,B)$ $(-1 \leq B < A \leq 1)$ of close-to-convex functions has been introduced. Certain integral transforms in this class have been considered. A necessary and sufficient condition in terms of convolutions for a function to be in $S_s^{*,n}(A,B)$ has been obtained. It is further shown that $S_s^{*,n}(A,B)$ is closed under convolution with the class of convex functions. Coefficient estimates for $S_s^{*,n}(A,B)$ and $S_s^{*,n}(\alpha)$ (with fixed $(n+1)$th coefficient) for f to be in the class $S_s^{*,n}(\alpha)$ have been obtained.

For any complex number μ, maximum value of the functional $|a_3-\mu a_2^2|$ for functions in this class has been determined. For different choices of parameters n,A and B, the results of this chapter yield along with some new results, the results of Goel and Mehrok [17], Silverman, Silvia and Telage [91], Silverman and Silvia [90] etc.

In Chapter III, the class $M(\lambda_1,\lambda_2)$ has been introduced. Using the technique of subordination, recently developed by S.S. Miller and P.T. Mocanu [55], it has been shown that for $\lambda_2 \leq 0$, functions in $M(\lambda_1,\lambda_2)$ are starlike and for $\lambda_1+\lambda_2 \geq 1$, functions in $M(\lambda_1,\lambda_2)$ are convex. An open problem that for $\lambda_2 > 0$ and $\lambda_1+\lambda_2 < 1$, the functions in $M(\lambda_1,\lambda_2)$ are starlike has been posed. Using the same technique, the subordination between certain class of functions has been obtained. The results of this chapter yield along with some new results, the results obtained by Miller and Mocanu [55], Mocanu and Reade [61] and others.

In Chapter IV, the order of starlikeness of a certain integral operator on certain subclasses of univalent and multivalent starlike functions have been obtained by using the sharp subordination results recently established by Miller and Mocanu [57] as well as a lemma due to D.R. Wilken and J. Feng [14]. The results of this chapter generalize

the results obtained by several authors such as Miller, Mocanu, Reade [59], Mocanu Reade and Repeanu [62], Ruscheweyh and V. Singh [82] .

Chapter V deals with the study of the effect of dropping first N-1 coefficients of Taylor expansion of a function about the origin on various properties of certains classes of analyti functions. The results of chapter yield along with some other results, the results obtained by Goel and Mehrok [16], Jakubowski and Kaminski [31], Juneja and Mogra [35] .

VI, the last chapter is devoted to the study of a certains integral operator on various subclasses of functions of the form $f(z) = \frac{1}{z} + a_0 + \sum_{n=1}^{\infty} a_n z^n$ regular in $0 < |z| < 1$. Some results of Goel and Sohi [18] have been corrected and generalized.

CHAPTER II

CERTAIN SUBCLASS OF CLOSE-TO-CONVEX FUNCTIONS

2.1 Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ belong to H and

(2.1.1) $f_n(z) = \frac{1}{n} \sum_{k=0}^{n-1} \omega^{n-k} f(\omega^k z)$

$$= z + a_{n+1} z^{n+1} + a_{2n+1} z^{2n+1} + \dots$$

where n is a positive integer and $\omega = e^{\frac{2\pi i}{n}}$ is the nth root of unity. We now introduce the class $S_s^{*,n}(A,B)$ as follows:

<u>Definition 2.1.1.</u> A function $f(z)$ in H is said to belong to $S_s^{*,n}(A,B)$ $(-1 \leq B < A \leq 1)$ if

(2.1.2) $$\frac{zf'(z)}{f_n(z)} \ll \frac{1+Az}{1+Bz}, \quad \text{for } z \in U .$$

If $f(z)$ is in $S_s^{*,n}(A,B)$, then from (2.1.2) we see that the conditions

(2.1.3) $$\left|\frac{zf'(z)}{f_n(z)} - \frac{1-AB}{1-B^2}\right| < \frac{A-B}{1-B^2}, \quad \text{for } z \in U \text{ if } B \neq -1$$

and

$$\operatorname{Re} \frac{zf'(z)}{f_n(z)} > \frac{1-A}{2}, \text{ for } z \in U \text{ if } B = -1$$

are also satisfied. $S_s^{*,n}(A,B)$ is the generalization of Sakaguchi's concept of function starlike with respect to

symmetric points. We further note that $S_s^{*,1}(A,B) = S^*(A,B)$. It can be easily shown that the functions in $S_s^{*,n}(A,B)$ are close-to-convex. Recently the class $S_s^{*,n}(1-2\alpha,-1) = S_s^{*,n}(\alpha)$ has been extensively studied by Rattan Chand [11], Prithvipal Singh [93] and T.R. Reddy [73]

In this chapter we first study the effect of a certain integral operator on functions in $S_s^{*,n}(A,B)$. Section 2.3 is devoted to study the effect of the inverse of above mentioned integral operator on functions in $S_s^{*,n}(A,B)$. In section 2.4 we obtain a necessary and sufficient condition in terms of convolutions for a function to be in $S_s^{*,n}(A,B)$. It has been shown that the class $S_s^{*,n}(A,B)$ is closed under convolution with the class of convex functions. In the last section of this chapter a coefficient inequality and coefficient estimates for functions in $S_s^{*,n}(A,B)$ have been obtained. We also determine the coefficient estimates for functions in $S_s^{*,n}(1-2\alpha,-1)$ with fixed (n+1)th coefficient. The results of this chapter yield along with some new results, the results of Goel and Mehrok [17], Silverman, Silvia and Telage [91], Silverman and Silvia [91], Rattan Chand [11] T.R. Reddy [75] etc.

2.2 In order to establish the results of this section, we first prove the following lemma.

<u>Lemma 2.2.1.</u> Let $\lambda(z)$ be a regular function defined in the unit disc U with Re $\lambda(z) > 0$ for $z \in U$. If $p(z)$ is

regular in U and $p(0) = 1$ with

(2.2.1) $\quad p(z) + [\lambda(z)\ zp'(z)] \ll \frac{1+Az}{1+Bz}$ for $z \in U$, $-1 \leq B < A \leq 1$

then

(2.2.2) $$p(z) \ll \frac{1+Az}{1+Bz}, \quad \text{for } z \in U.$$

Proof. Let

$$p(z) = \frac{1+A\omega(z)}{1+B\omega(z)} .$$

We have to show that if (2.2.1) is satisfied, then $|\omega(z)| < 1$ for $z \in U$. Suppose it is not true. Thus there exists a $z_o \in U$ such that $|\omega(z_o)| = 1$ and

(2.2.3) $$|\omega(z)| \leq |\omega(z_o)|, \quad \text{for } |z| \leq |z_o|.$$

Hence, by Jack's Lemma [30], there exists a $k \geq 1$ such that

(2.2.4) $$z_o\omega'(z_o) = k\,\omega(z_o)$$

In view of (2.2.3) and (2.2.4)

(2.2.5) $$p(z_o)+\lambda(z_o)z_op'(z_o) = \frac{1+A\omega(z_o)}{1+B\omega(z_o)} + \lambda(z_o)\ \frac{(A-B)k\omega(z_o)}{(1+B\omega(z_o))^2}\ ,\ k \geq 1$$

Let $\omega(z_o) = e^{it}$, $0 \leq t < 2\pi$. Then (2.2.5) reduces to

(2.2.6) $$p(z_o) + \lambda(z_o)z_op'(z_o) = \frac{1+Ae^{it}}{1+Be^{it}} + \frac{(A-B)ke^{it}}{(1+Be^{it})^2}\,\lambda(z_o)$$

whereas (2.2.1) gives

(2.2.7) $$\left|p(z_o) + \lambda(z_o)\ z_op'(z_o) - \frac{1-AB}{1-B^2}\right| < \frac{A-B}{1-B^2}, \text{ for } z_o \in U \text{ if } B \neq -1$$

and

$$(2.2.8) \qquad \mathrm{Re}\,\{p(z_o) + \lambda(z_o)z_op'(z_o)\} > \frac{1-A}{2}\,, \text{ for } z \in U \quad \text{if} \quad B = -1.$$

Application of (2.2.6) in (2.2.7) and (2.2.8) yields

$$(2.2.9) \qquad \left|\frac{B+e^{it}}{(1+Be^{it})(1-B^2)} + \frac{e^{it}k\lambda(z_o)}{(1+Be^{it})^2}\right| < \frac{1}{1-B^2}\,, \text{ for } z \in U \quad \text{if } B \neq -1$$

and

$$(2.2.10) \qquad \mathrm{Re}\,\left\{\frac{1+Ae^{it}}{1-e^{it}} + \frac{(A+1)ke^{it}}{(1-e^{it})^2}\,\lambda(z_o)\right\} > \frac{1-A}{2}, \text{ for } z \in U \quad \text{if} \quad B = -1.$$

Taking out $\dfrac{e^{it}(1+Be^{-it})}{(1+Be^{it})}$, whose modulus is 1, from the L.H.S. of (2.2.9), we get

$$\left|\frac{1}{1-B^2} + \frac{\lambda(z_o)k}{|1+Be^{it}|^2}\right| < \frac{1}{1-B^2}\,.$$

Also, (2.2.10) provides

$$\mathrm{Re}\,\left\{\frac{1-A}{2} - \frac{(A+1)}{4}\,\lambda(z_o)\right\} > \frac{1-A}{2}\,.$$

Both the above inequalities are impossible, consequently, $|w(z)| < 1$ for $z \in U$ and therefore,

$$p(z) \ll \frac{1+Az}{1+Bz}\,.$$

This completes the proof of the lemma. Now we will prove our main theorem.

<u>Theorem 2.2.1</u>. Let $f(z)$, analytic in the unit disc, belong to $S_s^{*,n}(A,B)$ $(-1 \leq B < A \leq 1)$. Consider the integral operator

$$I_c(f) = \frac{c+1}{c} \int_0^z f(w)\, w^{c-1} dw \;\; = F(z) \quad \text{say}.$$

If $\operatorname{Re} c > -\frac{1-A}{1-B}$, then $I_c(f)$ maps $S_s^{*,n}(A,B)$ into $S_s^{*,n}(A,B)$.

Proof. From the definition of $F(z)$, we have

$$(2.2.11) \qquad F(z) \left(\frac{zF'(z)}{F(z)} + c\right) = (c+1) f(z).$$

Also,

$$(2.2.12) \qquad F_n(z) \left(\frac{zF_n'(z)}{F_n(z)} + c\right) = (c+1)\, f_n(z).$$

Let

$$(2.2.13) \qquad \frac{zF'(z)}{F_n(z)} = p(z).$$

Differentiating both the sides of (2.2.11) and using (2.2.12) and (2.2.13), we get after a simple computation

$$(2.2.14) \qquad \frac{zf'(z)}{f_n(z)} = p(z) + zp'(z) \frac{1}{c + \frac{zF_n'(z)}{F_n(z)}} .$$

Also, (2.2.12) gives

$$(2.2.15) \qquad \frac{zF_n'(z)}{F_n(z)} + \frac{z\left(\frac{zF_n'(z)}{F_n(z)}\right)'}{\frac{zF_n'(z)}{F_n(z)} + c} = \frac{zf_n'(z)}{f_n(z)} .$$

Use of Lemma 2.2.1 in (2.2.15) provides that if $\operatorname{Re} c > -\frac{1-A}{1-B}$, then

$$\frac{zF_n'(z)}{F_n(z)} \ll \frac{1+Az}{1+Bz}, \quad \text{for } z \in U.$$

Hence,

$$\operatorname{Re}\left\{\frac{1}{\frac{zF_n'(z)}{F_n(z)}+c}\right\} > 0, \quad \text{for} \quad z \in U.$$

Since $f(z)$ belongs to $S_s^{*,n}(A,B)$, (2.2.14) provides

$$(2.2.16) \qquad p(z) + zp'(z)\,\frac{1}{c+\frac{zF_n'(z)}{F_n(z)}} \ll \frac{1+Az}{1+Bz}, \text{ for } z \in U.$$

Again, using Lemma 2.2.1 in (2.2.16), we get

$$p(z) \ll \frac{1+Az}{1+Bz} .$$

Hence, $F(z) \in S_s^{*,n}(A,B)$.

This completes the proof of the theorem.

Corollary 2.2.1 [17] Let $f(z) \in S_s^{*,2}(A,B)$ and

$$F(z) = \frac{2}{z}\int_0^z f(w)dw .$$

Then $F(z)$ also belongs to $S_s^{*,2}(A,B)$.

Corollary 2.2.2 [11] Let $f(z)$ belong to $S_s^{*,n}(\alpha)$ and

$$F(z) = \frac{c+1}{z^c}\int_0^z f(w)w^{c-1}dw \quad , \ c=1,2,3.... \quad .$$

Then $F(z) \in S_s^{*,n}(\alpha)$.

Corollary 2.2.3 [16] Let $f(z) \in S^*(A,B)$ and

$$F(z) = \frac{c+1}{z^c}\int_0^z f(w)w^{c-1}dw, \quad \operatorname{Re} c > -\frac{1-A}{1-B} .$$

Then $F(z) \in S^*(A,B)$.

Remark 2.2.1. According to Lewandowski and J. Stankiewicz [41] two normalized functions $f(z)$ and $g(z)$, both regular in U, are called mutually adjoint if

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)+g(z)}\right\} > 0 \quad \text{and} \quad \operatorname{Re}\left\{\frac{zg'(z)}{f(z)+g(z)}\right\} > 0, \quad z \in U.$$

One can easily see that if $f(z)$ and $g(z)$ are mutually adjoint, then $\varphi(z) = \frac{f(z)+g(z)}{2}$ belong to S^* and both $f(z)$ and $g(z)$ are close-to-convex.

Further, let $f_1, f_2, \ldots f_n$ be normalized functions regular in U. We say that $\{f_1, f_2, \ldots f_n\} \in \Phi_n(A,B)$ $(-1 \leq B < A \leq 1)$ if for $z \in U$

$$\left\{\frac{nzf_i'(z)}{\sum_{i=1}^{n} f_i(z)}\right\} \ll \frac{1+Az}{1+Bz}, \quad \forall\, i = 1,2,\ldots n.$$

One can easily check that $\varphi(z) = \frac{\sum_{i=1}^{n} f_i(z)}{n}$ belongs to $S^*(A,B)$ and each f_i is close-to-convex. Following the lines of proof of Theorem 2.2.1, we can prove

Theorem 2.2.2. Let $f_1, f_2 \ldots f_n$ be normalized regular functions in U and $\{f_1, f_2 \ldots f_n\} \in \Phi_n(A,B)$ $(-1 \leq B < A \leq 1)$. If $\operatorname{Re} c > -\frac{1-A}{1-B}$ and

$$F_i(z) = \frac{c+1}{z^c} \int_0^z f_i(w) w^{c-1}\, dw, \quad i=1,2,\ldots n,$$

then $\{F_1, F_2, \ldots F_n\} \in \Phi_n(A,B)$.

Corollary 2.2.3. Let $f(z)$ and $g(z)$ be mutually adjoint. If

$$F(z) = \frac{c+1}{z^c}\int_0^z f(w)w^{c-1}dw \quad \text{and} \quad G(z) = \frac{c+1}{z^c}\int_0^z g(w)w^{c-1}dw, \ \mathrm{Re}\ c>0$$

then $F(z)$ and $G(z)$ are also mutually adjoint.

2.3 Let $f(z)$ be analytic in the unit disc U. If

$$F(z) = \frac{c+1}{z^c}\int_0^z f(w)w^{c-1}dw \ , \ \mathrm{Re}\ c > -1,$$

then one can easily check that

$$f(z) = \frac{F(z)}{c+1}\left(\frac{zF'(z)}{F(z)} + c\right) = F(z)*h_c(z)$$

where

$$(2.3.1) \qquad h_c(z) = \frac{z - \frac{c}{c+1}z^2}{(1-z)^2} \quad .$$

In the previous section we have shown that $F(z) \in S_s^{*,n}(A,B)$, whenever $f(z) \in S_s^{*,n}(A,B)$ and $\mathrm{Re}\ c > -\{\frac{1-A}{1-B}\}$. When $F(z) \in S^*(A,B)$, what can be the maximum value of r, $(0 < r < 1)$ such that for $|z| \le r$

$$\left|\frac{zf'(z)}{f_n(z)} - \frac{1-AB}{1-B^2}\right| < \frac{A-B}{1-B^2}\ , \quad \text{if} \quad B \neq -1$$

and

$$\mathrm{Re}\left\{\frac{zf'(z)}{f_n(z)}\right\} > \frac{1-A}{2}\ , \quad \text{if} \quad B = -1$$

is the purpose of this section. To some extent, it can be termed as converse to Theorem 2.2.1.

For solving this problem we need the following lemmas:

Lemma 2.3.1 [4] Let φ and g be analytic in U with $\varphi(0) = g(0) = 0$ and $\varphi'(0)\, g'(0) \neq 0$. Suppose for each $\alpha(|\alpha| = 1)$ and $\sigma(|\sigma| = 1)$ we have

$$[\,\varphi * \frac{1+\alpha\sigma z}{1-\sigma z}\, g] \neq 0 \quad \text{for} \quad 0 < |z| < r < 1.$$

Then for each F analytic in U, the image of $|z| < r$ under $(\varphi * Fg)/(\varphi * g)$ is a subset of the convex hull of F(U).

Lemma 2.3.2 [4] Let $f_\alpha(z) = \frac{(1+\alpha)z}{(1-z)(1+\alpha z)}$ $(|\alpha|=1)$. If $|z| \leq r < 1$, then

$$|f_\alpha(z)| \leq \frac{2r}{1-r^2}\,.$$

We are now in a position to prove the following theorem.

Theorem 2.3.1. Let $F(z)$ belong to $S_s^{*,n}(A,B)$ and

$$(2.3.2) \qquad f(z) = \frac{F(z)}{c+1}\left(\frac{zF'(z)}{F(z)} + c\right), \qquad \operatorname{Re} c > -1.$$

If r is such that

$$(2.3.3) \qquad \operatorname{Re} c \geq \frac{2r}{1-r^2} - \frac{1-Ar^n}{1-Br^n}\,,$$

then for $|z| < r$

$$(2.3.4) \qquad \left|\frac{zf'(z)}{f_n(z)} - \frac{1-AB}{1-B^2}\right| < \frac{A-B}{1-B^2}\,, \text{ if } B \neq -1$$

and

$$(2.3.5) \qquad \operatorname{Re} \frac{zf'(z)}{f_n(z)} > \frac{1-A}{2} \quad \text{if} \quad B = -1.$$

Proof. Since $F(z)$ belongs to $S_s^{*,n}(A,B)$

$$\frac{zF'(z)}{F_n(z)} = H(z) \ll \frac{1+Az}{1+Bz} .$$

We have to find maximum $r (0 < r < 1)$ such that for $|z| < r$

$$\left|\frac{zf'(z)}{f_n(z)} - \frac{1-AB}{1-B^2}\right| < \frac{A-B}{1-B^2}, \quad \text{if } B \neq -1$$

and

$$\operatorname{Re} \frac{zf'(z)}{f_n(z)} > \frac{1-A}{2}, \text{ if } B = -1.$$

Since

$$\frac{zf'(z)}{f_n(z)} = \frac{z(F(z)*h_c(z))'}{F_n(z)*h_c(z)} = \frac{zF'(z)*h_c(z)}{F_n(z)*h_c(z)} = \frac{F_nH*h_c(z)}{F_n(z)*h_c(z)} ,$$

where $h_c(z)$ is defined in (2.3.1), by lemma 2.3.1, it is sufficient to show that

$$[h_c(z)* \frac{1+\alpha\sigma z}{1-\sigma z} F_n(z)] \neq 0 \text{ for } 0 < |z| < r < 1.$$

Let

$$h_c(z)* \frac{1+\alpha\sigma z}{1-\sigma z} F_n(z) = G(z).$$

Then

$$(2.3.6) \quad (1+c)G(z) = \frac{1+\alpha\sigma z}{1-\sigma z} F_n(z)\left[c + \frac{zF_n'(z)}{F_n(z)} + \frac{(1+\alpha)\sigma z}{(1-\sigma z)(1+\alpha\sigma z)}\right].$$

Since $F(z)$ belongs to $S_s^{*,n}(A,B)$, therefore $F_n(z) \neq 0$ in $0 < |z| < 1$ and

$$(2.3.7) \quad \operatorname{Re} \frac{zF_n'(z)}{F_n(z)} \geq \frac{1-Ar^n}{1-Br^n}, \quad \text{for } |z| \leq r.$$

Also, Lemma 2.3.2 provides

(2.3.8) $\left|\frac{(1+\alpha)\,\sigma z}{(1-\sigma z)(1+\alpha\sigma z)}\right| \leq \frac{2r}{1-r^2}$, for $|z| \leq r$.

Using the inequalities (2.3.7) and (2.3.8) in (2.3.6) one can easily claim that $G(z) \neq 0$ for $|z| < r$ and consequently, in view of Lemma 2.3.1 the inequalities (2.3.4) and (2.3.5) are satisfied for $|z| < r$.

This completes the proof of the theorem.

Remark 2.3.1. We are not able to claim the sharpness of our result in case $B \neq -1$. For $B = -1$, the function

$$F(z) = \int_0^z \frac{1+A\rho}{(1-\rho)(1-\rho^n)^{\frac{1+A}{n}}}\, d\rho$$

gives the sharp result for n odd

Corollary 2.3.1. Let $F(z)$ be starlike of order α and

$$f(z) = \frac{F(z)}{c+1}\left(\frac{zF'(z)}{F(z)} + c\right), \quad \text{Re } c > -1.$$

If r is such that

$$\text{Re } c \geq \frac{2r}{1-r^2} - \frac{1-(1-2\alpha)r}{1+r},$$

then

$$\text{Re } \frac{zf'(z)}{f(z)} > \alpha.$$

2.4 In this section we first obtain a necessary and sufficient condition for a function $f(z)$ to belong to $S_s^{*,n}(A,B)$. Also, we show that $S_s^{*,n}(A,B)$ is closed under convolution with the class of convex functions.

Theorem 2.4.1. A function $f(z)$ in H belongs to $S_s^{*,n}(A,B)$ if and only if it satisfies

$$\frac{1}{z}\left[f(z)* \frac{z+\frac{z^2}{B-A}\left[(x+A)+(B+x)\left(\frac{1-z^{n-1}}{1-z}\right)\right]}{(1-z)(1-z^{n-1})}\right] \neq 0$$

for $z \in U$, $|x| = 1$.

Proof. A function $f(z)$ in H belongs to $S_s^{*,n}(A,B)$ if and only if

$$\frac{zf'(z)}{f_n(z)} \ll \frac{1+Ax}{1+Bx}, \quad z \in U.$$

Hence,

$$(2.4.2) \qquad \frac{zf'(z)}{f_n(z)} \neq \frac{1+Ax}{1+Bx}, \text{ for } z \in U \text{ and } |x| = 1$$

or

$$\frac{1}{z}\left[(1+Bx)zf'(z) - f_n(z)(1+Ax)\right] \neq 0 \text{ for } z \in U \text{ and } |x| = 1.$$

Since

$$zf'(z) = f(z)* \frac{z}{(1-z)^2} \text{ and } f_n(z) = f(z)* \frac{z}{(1-z^n)},$$

the left hand side (2.4.3) becomes

$$(2.4.3) \quad \frac{1}{z}\left[(1+Bx)f(z)* \frac{z}{(1-z)^2} - f(z)* \frac{z}{(1-z^n)}(1+Ax)\right]$$

which on simplification reduces to

$$\frac{1}{z}\left[f(z)* \frac{(B-A)zx+z^2\left[(1+Ax)+(1+Bx)\left(\frac{1-z^{n-1}}{1-z}\right)\right]}{(1-z)(1-z^n)}\right]$$

Hence, (2.4.3) is equivalent to

$$\frac{1}{z}\left[f(z) * \frac{z + \frac{z^2}{B-A}\left[(x+A) + (B+x)\left(\frac{1-z^{n-1}}{1-z}\right)\right]}{(1-z)(1-z^n)}\right] \neq 0$$

for $z \in U$ and $|x| = 1$,

which is the desired condition.

For $n=1$, $A=1-2\alpha$ and $B = -1$, we obtain the following convolution condition for a function $f(z)$ to be in $S^*(\alpha)$ which is due to Silverman, Telage and Silvia [91] .

Corollary 2.4.1. A function $f(z)$ in H is starlike of order α if and only if

(2.4.4) $$\frac{1}{z}\left[f(z) * \frac{z + \frac{x+(2\alpha-1)}{2(1-\alpha)}}{(1-z)^2}\right] \neq 0, \text{ for } z \in U \text{ and } |x| = 1.$$

Corollary 2.4.2. A function $f(z)$ in H belong to $S^*(A,B)$ if and only if

(2.4.5) $$\frac{1}{z}\left[f(z) * \frac{z + \frac{z^2(A+x)}{B-A}}{(1-z)^2}\right] \neq 0 \text{ for } z \in U.$$

Theorem 2.4.2. If $\varphi \in K$ and f belongs to $S_s^{*,n}(A,B)$, then $\varphi * f$ belongs to $S_s^{*,n}(A,B)$.

Proof. Let

$$\frac{zf'(z)}{f_n(z)} = h(z) \qquad \text{(say)}.$$

Since $f(z)$ belongs to $S_s^{*,n}(A,B)$, $f_n(z) \in S^*(A,B)$ and

$$h(z) \prec\prec \frac{1+Az}{1+Bz},$$

we have to show that

$$\frac{z(\varphi*f)'}{(\varphi*f)_n} \prec\prec \frac{1+Az}{1+Bz} .$$

Since

$$\frac{z(\varphi*f)'}{(\varphi*f)_n} = \frac{\varphi*zf'}{\varphi*f_n} = \frac{\varphi*f_n h}{\varphi*f_n} ,$$

by lemma (2.5.1) it is sufficient to show that

(2.4.6) $\varphi*(\frac{1+\alpha\sigma z}{1-\sigma z})f_n \neq 0$ for $0 < |z| < 1$, $|\alpha| = 1$ and $|\sigma|=1$

Since $f_n \in S^*(A,B) \subset S^*$, (2.4.6) is true due to Ruschewey.
Hence, $\varphi*f \in S_s^{*,n}(A,B)$.

This completes the proof of the theorem.

5. We now obtain some coefficient inequalities and sharp coefficient estimates for $f(z) \in S_s^{*,n}(A,B)$, $(-1 \leq B < A \leq 1)$. Also, we obtain coefficient estimates for $f(z) = z+ \sum_{p=2}^{\infty} a_p z^p$ belonging to $S_s^{*,n}(\alpha)$ with fixed a_{n+1} such that $|a_{n+1}| = a \leq \frac{2(1-\alpha)}{n}$.

Theorem 5.1. Let $f(z) = z + \sum_{n=2}^{\infty} b_k z^k$, then for n=1

(2.5.1) $|b_3-ub_2^2| \leq \frac{A-B}{2} \max \{1, |2B- A + 2u(A-B)|\}$

(2.5.2) $|b_3-ub_2^2| \leq \frac{A-B}{2} \max \{1, |B+ \frac{1}{2}(A-B)u|\}$, for n=2

(2.5.3) $|b_3-ub_2^2| \leq \frac{A-B}{3} \max \{1, |B+ \frac{3}{4}(A-B)u|\}$, for $n \geq 3$.

Proof. Since $f(z)$ belongs to $S_s^{*,n}(A,B)$,

$$(2.5.4) \qquad \frac{zf'(z)}{f_n(z)} = \frac{1+A\omega(z)}{1+B\omega(z)}$$

where $\omega(z)$ is a schwarz function.

Let $\omega(z) = \sum_{k=1}^{\infty} \beta_k z^k$. Equating the coefficients of both the sides of (2.5.4), we get

$$\beta_1 = \frac{b_2}{A-B}$$

and

$$\beta_2 = \frac{2}{A-B}\left[b_3 - \frac{b_2^2}{2(A-B)}(A-2B)\right] \text{ , for } n = 1.$$

For n=2,

$$\beta_1 = \frac{2b_2}{A-B}$$

$$\beta_2 = \frac{2}{A-B}\left[b_3 + \frac{2Bb_2^2}{A-B}\right]$$

and for $n \geq 3$,

$$\beta_1 = \frac{2b_2}{A-B}$$

$$\beta_2 = \frac{3}{A-B}\left[b_3 + \frac{4Bb_2^2}{A-B}\right] .$$

Using lemma of Keogh and Merkes [38], we have

$$(2.5.5) \qquad |\beta_2 - v\beta_1^2| \leq \max\{1,|v|\} \text{ , for any complex number } v.$$

Hence, for n=1,

$$(2.5.6) \qquad |b_3 - ub_2^2| = \frac{A-B}{2}\left[\beta_2 - v\beta_1^2\right]$$

where $v = 2u(A-B)-2(A-2B)$.

For $n = 2$

$$|b_3 - ub_2^2| = \frac{A-B}{2}\,[\beta_2 - v\beta_1^2]$$

where $v = B + \frac{1}{2}u(A-B)$.

For $n \geq 3$

$$(2.5.8) \qquad |b_3 - ub_2^2| = \frac{A-B}{2}\,[\beta_2 - v\beta_1^2]$$

where $v = B + \frac{3}{4}(A-B)u$.

Using (2.5.5) in (2.5.6), (2.5.7) and (2.5.8), we get (2.5.1), (2.5.2) and (2.5.3) respectively. For $n=1,2$, the results are due to Goel and Mehrok [16],[17] respectively.

In the following theorems we obtain the coefficient estimates for a function in $S_s^{*,n}(A,B)$. First, we obtain the coefficient estimates for $f(z) \in S_s^{*,n}(A,B)$ where A and B are such that $A-(n+1)B \leq n$.

<u>Theorem 2.5.2</u>. Let $f(z) = z + \sum_{p=2}^{\infty} a_p z^p$ belong to $S_s^{*,n}(A,B)$ $(-1 \leq B < A \leq 1)$. If $A-(n+1)B \leq n$, then

$$(2.5.9) \qquad |a_p| \leq \frac{A-B}{p}, \quad \text{for } p \neq nk+1,\ k \geq 1$$

and

$$(2.5.10) \qquad |a_{nk+1}| \leq \frac{A-B}{nk}, \quad \text{for } k \geq 1.$$

The bounds in (2.5.9) are sharp for the function

$$(2.5.11) \qquad f(p,z) = \int_0^z \frac{1+A\rho^p}{1+B\rho^p} \cdot \exp \int_0^\rho \left(\frac{\varphi(t)-1}{t}\right) dt\, d\rho$$

where $$\varphi(z) = \frac{1+Az^p}{1+Bz^p} * \frac{1}{1-z^n} .$$

For $B=0$, the function in (2.5.11) reduces to

$$f(p,z) = z + \frac{Az^p}{p}$$

The bounds in (2.5.10) are sharp for the functions

$$(2.5.12) \quad f(n_{k+1},z) = \begin{cases} z(1+Bxz^{nk})^{\frac{A-B}{nkB}} & , \text{ for } B \neq 0, \\ z\exp\left(\frac{Axz^{nk}}{nk}\right) & , \text{ for } B=0, \ |x| = 1. \end{cases}$$

Proof. Since $f(z)$ belongs to $S_s^{*,n}(A,B)$,

$$\frac{zf'(z)}{f_n(z)} = \frac{1+A\omega(z)}{1+B\omega(z)} , \quad (\omega(z) \text{ is a schwarz function})$$

i.e.

$$(2.5.13) \qquad [zf'(z) - f_n(z)] = \omega(z)[Af_n(z)-Bzf'(z)]$$

Feeding the valuc of $zf'(z)$ and $f_n(z)$ in (2.5.13), we get

$$\sum_{m=2}^{\infty} (m-\delta_m)\, a_m z^{m-1} = \omega(z) \left[(A-B) + \sum_{m=2}^{\infty} (A\delta_m-mB)a_m z^{m-1}\right]$$

where

$$\delta_m = \begin{cases} 1, & \text{if } m=nk+1 \text{ for some } k \geq 1, \\ 0, & \text{if there does not exist any } k \geq 1 \text{ such that} \\ & \qquad m = nk+1 \end{cases}$$

Setting

$$s_p(z) = \sum_{m=2}^{p} (m-\delta_m)\, a_m z^{m-1}$$

and

$$S_p(z) = (A-B) + \sum_{m=2}^{p-1} (A\delta_m - mB) z^{m-1} ,$$

we have

$$(2.5.14) \quad s_p(z) + \sum_{m=p+1}^{\infty} (m-\delta_m) a_m z^{m-1}$$

$$= \omega(z) \left[S_p(z) + \sum_{m=p}^{\infty} (A\delta_m - mB)\, a_m z^{m-1} \right]$$

due to (2.5.13).

Since $|\omega(z)| < 1$ for $|z| \leq r < 1$,

$$\sum_{m=2}^{p} (m-\delta_m)^2 |a_m|^2 r^{2m-2} \leq (A-B)^2 + \sum_{m=2}^{p-1} (A\delta_m - mB)^2 |a_m|^2 r^{2m-2}.$$

Letting $r \to 1$, we get

$$(2.5.15) \quad \sum_{m=2}^{p} (m-\delta_m)^2 |a_m|^2 \leq (A-B)^2 + \sum_{m=2}^{p-1} [(A\delta_m - mB)^2 - (m-\delta_m)^2]\, |a_m|^2.$$

If $n=1$ and $A-2B \leq 1$, then one can easily see that

$$|a_p| \leq \frac{A-B}{p-1}.$$

Now, $|mB| \leq m$ is obviously true for $-1 \leq B \leq 1$. Further, if $m=nk+1$ for some $k \geq 1$, then by the conditions given in the theorem, we have

$$|A-(nk+1)B| \leq nk \qquad \text{for} \quad k \geq 1.$$

Using these inequalities in (2.5.15), we get

$$|p-\delta_p|^2 |a_p|^2 \leq (A-B)^2$$

(2.5.9) and (2.5.10) follow easily from the above inequality. This completes the proof of the theorem.

Now we will obtain the coefficient estimates for $f(z) \in S_s^{*,n}(A,B)$, where $A-B(n+1) > n$. For that we need the following lemma

Lemma 2.5.1. If m is a natural number such that $m \geq 2$, then

$$\frac{1}{(mn)^2}\{(A-B)^2 + \sum_{k=1}^{m-1}[(A-(nk+1)B)^2-(nk)^2]\prod_{j=0}^{k-1} u_j\} = \prod_{j=0}^{m-1} u_j$$

(2.5.16) where $u_j = \dfrac{(A-(jn+1)B)^2}{n^2(j+1)^2}$, $j=0,1,2....$ and n is fixed.

Proof. We prove the lemma by induction on m. For $m=1$, it is obviously true. Let us suppose that it is true for $m=p-1$, $p\geq 3$. Then for $m=p$, the left hand side of (2.5.16) reduces to

$$\therefore \frac{1}{(pn)^2}\{(A-B)^2+\sum_{k=1}^{p-2}[(A-(nk+1)B)^2-(nk)^2]\prod_{j=0}^{k-1}u_j\}$$

$$+[(A-(n(p-1)+1)B)^2-n^2(p-1)^2]\prod_{j=0}^{p-2}u_j$$

$$=\frac{1}{(pn)^2}[(n^2(p-1)^2\prod_{j=0}^{p-2}u_j+\{(A-(n(p-1)+1)B)^2-n^2(p-1)^2\}\times\prod_{j=0}^{p-2}u_j$$

$$=\frac{1}{(pn)^2}[(A-(n(p-1)+1)B)^2-n^2(p-1)^2]\prod_{j=0}^{p-2}u_j$$

$$=\prod_{j=0}^{p-1}u_j \ .$$

This completes the proof of the lemma.

Theorem 2.5.3. Let $f(z)\in S_s^{*,n}(A,B)$ $(-1\le B<A\le 1)$ and K be the smallest positive integer such that $A-B(nK+1)\le nK$. Then

$$(2.5.17)\qquad |a_p|\le\frac{kn}{p}\left(\prod_{j=0}^{k-1}u_j\right)^{1/2}\quad\text{for } (k-1)n+2\le p\le kn$$

and

$$(2.5.18)\qquad |a_p|\le\left(\prod_{j=0}^{k-1}u_j\right)^{1/2}\qquad\text{for } p=nk+1$$

where $k\le K$.

If $p\ge nK+2$, then

$$(2.5.19)\qquad |a_p|\le\frac{nK}{(p-\delta_p)}\left(\prod_{j=0}^{K-1}u_j\right)^{1/2}$$

where u_j is as given in (2.5.16).

Proof. The theorem follows easily by using Lemma 2.5.1 in (2.5.15).

In the following theorem we obtain the coefficient estimates for $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ belonging to $S_s^{*,n}(1-2\alpha,-1)$ where $|a_{n+1}| = a \leq \frac{2(1-\alpha)}{n}$.

Theorem 2.5.14. Let $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ belong to $S_s^{*,n}(1-2\alpha,-1) = S_s^{*,n}(\alpha)$ and $|a_{n+1}| = a \leq \frac{2(1-\alpha)}{n}$. Then

$$(2.5.20) \qquad |a_p| \leq \frac{2(1-\alpha)}{p}, \quad \text{for } 2 \leq p \leq n,$$

$$(2.5.21) \qquad |a_p| \leq \frac{1+a}{(p-1)(n+2-2\alpha)} \frac{\prod_{j=0}^{k} (nj+2-2\alpha)}{\lfloor k \; n^{k-1}},$$

for $p = n(k+1) + 1$, $k \geq 1$,

and

$$(2.5.22) \qquad |a_p| \leq \frac{1+a}{p(n+2-2\alpha)} \frac{\prod_{j=0}^{k} (nj+2-2\alpha)}{\lfloor k \; n^{k-1}},$$

for $nk+2 \leq p \leq n(k+1)$.

Proof. Since $f(z) \in S_s^{*,n}(1-2\alpha -1)$, there exists a schwarz function $\omega(z)$ such that

$$(2.5.23) \qquad \frac{zf'(z)}{f_n(z)} = \frac{1+(1-2\alpha)\omega(z)}{1-\omega(z)} = 1+ \sum_{m=1}^{\infty} c_m z^m = p(z) \quad \text{(say)}$$

i.e.

$$(2.5.24) \qquad (z + \sum_{k=2}^{\infty} ka_k z^k) = (1+ \sum_{m=1}^{\infty} c_m z^m)(z + \sum_{k=1}^{\infty} a_{nk+1} z^{nk+1})$$

(2.5.23) provides that $\operatorname{Re} p(z) > \alpha$ for $z \in U$ and, therefore,

$$|c_n| \leq 2(1-\alpha) \quad \text{for} \quad n \geq 1.$$

Equating the coefficients of both the sides of (2.5.23), we get

$$(2.5.25) \qquad p\, a_p = \sum_{m=0}^{k} c_{p-mn+1}\, a_{mn+1} \qquad (a_1=1)$$

for $nk+2 \leq p \leq n(k+1)$ and for $p = n(k+1)+1$

$$pa_p = a_p + \sum_{m=0}^{k} c_{p-mn+1}\, a_{mn+1}$$

i.e.,

$$(2.5.26) \qquad (p-1)a_p = \sum_{m=0}^{k} c_{p-mn+1}\, a_{mn+1}$$

(2.5.20) follows easily from (2.5.25) and the fact that $|c_n| \leq 2(1-\alpha)$. We will prove (2.5.21) and (2.5.22) by induction. One can easily see that it is true for $n+2 \leq p \leq 2n+1$. We will show that the inequality (2.5.21) is true for $p=nk+2$, whenever (2.5.21) and (2.5.22) are true for $n+2 \leq p \leq kn+1$ (2.5.25) provides

$$(kn+1)|a_{kn+2}| \leq (2-2\alpha)\ [1+|a_{n+1}|+\ldots (a_{kn+1}|]$$

$$= (2-2\alpha)[(1+a)+\frac{1+a}{n+2-2\alpha}\sum_{m=1}^{k-1}\frac{\prod_{j=0}^{m}(jn+2-2\alpha)}{n^m\ \lfloor(m+1)}]$$

$$= (2-2\alpha)[(1+a)+\frac{1+a}{n+2-2\alpha}\{\frac{\prod_{j=0}^{k}(jn+2-2\alpha)}{n^{k-1}\ \lfloor k} - (n+2-2\alpha)\}]$$

$$= (2-2\alpha)\ [\frac{\prod_{j=0}^{k}(jn+2-2\alpha)}{n^{k-1}\ \lfloor k}]\ \frac{1+a}{n+2-2\alpha}$$

Hence,

$$|a_{nk+2}| \leq \frac{1+a}{(n+2-2\alpha)(kn+1)}\ \frac{\prod_{j=0}^{k}(jn+2-2\alpha)}{n^{k-1}\ \lfloor k} \tag{2.5.27}$$

(2.5.21) and (2.5.22) follow easily by using (2.5.27) in (2.5.25) and (2.5.28).

For n=1 it reduces to the result of Silverman and Silvia [90]

<u>Corollary 2.5.1</u> [90] Let $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$ and $|a_2| = a \leq 2(1-\alpha)$.

Then

$$|a_j| \leq \frac{1+a}{3-2\alpha}\ \frac{\prod_{j=2}^{k}(k-2\alpha)}{\lfloor(j-1)} \qquad j = 3,4,\ldots .$$

CHAPTER III

SUBORDINATION BETWEEN CERTAIN CLASSES OF FUNCTIONS

3.1 In this chapter we first introduce the following subclass of H.

Definition 3.1.1. Let $M(\lambda_1,\lambda_2)$ denote the class of functions $f(z)$ in H such that

$$\frac{f.g.h}{z^3} \neq 0 \text{ and}$$

$$\text{Re}\,\{(1-\lambda_1-\lambda_2)\,\frac{zf'}{f} + \lambda_1\,\frac{zg'}{g} + \lambda_2\,\frac{zh'}{h}\} > 0\ , \quad \text{for } z \in U,$$

$$-\infty < \lambda_i < \infty,\ i = 1,2.$$

where $zf' = g$ and $zg' = h$.

Mocanu [60] introduced the class of λ-convex functions $M(\lambda)$, $0 \leq \lambda \leq 1$. Later, it was generalized for all $(-\infty < \lambda < \infty)$. It is well known that $M(\lambda) \subseteq S^*$ for $-\infty < \lambda < \infty$. $M(\lambda)$ can be considered as having the linear combination of starlikeness and convexity properties. A lot of work has been done on the Mocanu class [54, 58, 59] . Since $M(\lambda_1,0) = M(\lambda_1)$, $M(\lambda_1,\lambda_2)$ is a generalization of Mocanu class.

Recently S.S. Miller and P.T. Mocanu [55] developed the technique of differential subordination. Using their technique, we are able to prove that for $\lambda_2 \leq 0$, $M(\lambda_1,\lambda_2) \subseteq S^*$. Also, we show that for $\lambda_1 + \lambda_2 \geq 1$, $M(\lambda_1,\lambda_2)$ is contained in the

class of convex functions. We obtain some containment relations for $M(\lambda_1,\lambda_2)$ depending on λ_1 and λ_2. Using their technique we obtain the subordination of a certain class of functions having initial gap to functions having symmetrical gaps.

Let $S_N^*(\alpha)$ $(\alpha < 1)$ denote the class of functions $f(z)$ belonging to $S^*(\alpha)$ $(\alpha < 1)$ and having the initial gap of width N. We obtain the order of starlikness of the integral operator

$$g(z) = \left(\frac{\beta+\gamma}{z^\gamma}\int_0^z f^\beta(w)w^{\gamma-1}dw\right)^{1/\beta}$$

where $f(z)$ belongs to $S_N^*(\alpha)$. As a corollary we obtain the order of starlikness of f belonging to $M(\lambda)$ having initial missing coefficients. Also, we show that the Libera integral operator maps the starlike functions of order-1 having the vanishing second coefficient into starlike functions. The results of this chapter yield along with some new results, the results obtained by Miller and Mocanu [56] , Mocanu and Read [61] etc.

3.2 In this section we first give some definitions and theorems due to Miller and Mocanu [56] , which are further needed in proving our results.

Definition 3.2.1. A function $q \in Q$, if it is regular and univalent on $\bar{U}$ (closure of U) except for those points $\xi \in \partial U$ for which $\lim\limits_{z \to \xi} q(z) = \infty$.

Definition 3.2.2. Let Ω be a domain in C and $q \in Q$. We define $\Psi_n(\Omega,q)$ to be the class of functions $\psi: C^3 \to C$ that satisfy the following:

(a) $\psi(r,s,t)$ is continuous in a domain $D \subset C^3$.

(b) $(q(0),0,0) \in D$ and $\psi(q(0),0,0) \in \Omega$.

(c) $\psi(r_o,s_o,t_o) \notin \Omega$ when $(r_o,s_o,t_o) \in D$ where $r_o = q(\xi)$, $s_o = m\xi q'(\xi)$ and

$$\text{Re}\left[1+\frac{t_o}{s_o}\right] \geq m\,\text{Re}\left[1+\frac{\xi q''(\xi)}{q'(\xi)}\right]$$

$|\xi|=1$, $q(\xi)$ is finite and $m \geq n \geq 1$. $\Psi_1(\Omega,q) \equiv \Psi(\Omega,q)$.

Theorem 3.2.1. Let $q(0) = a$ and let $\psi \in \Psi_n$ with corresponding domain D. Let $p(z) = a+p_nz^n + p_{n+1}z^{n+1}+\dots$ be regular in U with $p(0) \not\equiv a$ and $n \geq 1$. If $(p(z), zp'(z), z^2p''(z)) \in D$ when $z \in U$ and

$$\psi(p(z), zp'(z), z^2p''(z)) \in \Omega \quad \text{when} \quad z \in U,$$

then $p(z) \ll q(z)$.

Definition 3.2.3. Let h be a conformal mapping of U onto Ω and $q \in Q$. Then $\Psi_n(h,q)$ denotes the class of functions $\psi \in \Psi_n(\Omega,q) = \Psi_n(h(U),q)$ which are holomorphic in their corresponding domains D and $\psi(q(0),0) = h(0)$.

Theorem 3.2.2. Let $\psi \in \Psi_n(h,q)$ with corresponding domain D and with $q(0) = a$. Let $p(z) = a+p_nz^n+p_{n+1}z^{n+1}+\dots$ be regular in U with $p(z) \not\equiv a$ and $n \geq 1$.

If

$$(p(z),\ zp'(z),\ z^2p''(z)) \in D \quad \text{when} \quad z \in U,$$

then

$$\psi(p(z),\ zp'(z),\ z^2p''(z)) \ll h(z) \text{ implies } p(z) \ll g(z).$$

On checking the definitions of $q(z)$ and $\Psi_n(\Omega,q)$, we see that the hypothesis of Theorem 3.2.1 and Theorem 3.2.2 require that $q(z)$ should behave nicely on ∂U. If this is not the case or if the behaviour of $q(z)$ on ∂U is not known, it is still possible to prove that $p(z) \ll q(z)$ by limiting procedure as given in the Theorem 3.2.3 and Theorem 3.2.4.

Theorem 3.2.3. Let $q(z)$ be univalent in U with $q(0) = a$ and let $q_\rho(z) = q(\rho z)$ for $0 < \rho < 1$. Let $\psi \in \Psi_n(\Omega, q_\rho)$ with domain D, for $0 < \rho < 1$ and let $p(z) = a+p_nz^n+p_{n+1}z^{n+1}+\dots$ be regular in U with $p(z) \not\equiv a$ and $n \geq 1$. If $(p(z),\ zp'(z),\ z^2p''(z)) \in D$ when $z \in U$ and $\psi(p(z),\ zp'(z),\ z^2p''(z)) \in \Omega$ when $z \in U$, then $p(z) \ll q(z)$.

Theorem 3.2.4. Let $h(z)$ and $q(z)$ be univalent in U with $q(0) = a$ and $h_\rho(z) = h(\rho z)$, $q_\rho(z) = q(\rho z)$, for $0 < \rho < 1$. Let $\psi \in \Psi_n(h_\rho, q_\rho)$ with domain D, for $0 < \rho < 1$ and let $p(z) = a+p_nz^n+p_{n+1}z^{n+1}+\dots$ be regular in U with $p(z) \not\equiv a$ and $n \geq 1$. If

$$(p(z),\ zp'(z),\ z^2p''(z)) \in D \quad \text{when} \quad z \in U,$$

then

$\psi(p(z), zp'(z), z^2p''(z)) \ll h(z)$ implies $p(z) \ll q(z)$.

We are now in a position to prove the following theorem:

Theorem 3.2.5. Let f belong to $M(\lambda_1,\lambda_2)$. If $\lambda_2 \leq 0$, then $f \in S^*$ and $f \in K$ if $\lambda_1 + \lambda_2 \geq 1$.

Proof. Let $\lambda_2 \leq 0$ and

$$p(z) = \frac{zf'(z)}{f(z)} = r,$$

$$zp'(z) = s \quad \text{and} \quad z^2p''(z) = t.$$

After a simple computation, we get

$$(3.2.1) \quad (1-\lambda_1-\lambda_2)\frac{zf'}{f} + \lambda_1\frac{zg'}{g} + \lambda_2\frac{zh'}{h} = r+\lambda_1\left(\frac{s}{r}\right)+\lambda_2\left(\frac{2rs+t+s}{r^2+s}\right)$$

where $g = zf'$ and $h = zg'$.

Let $\psi: C^3 \to C$ be given by

$$(3.2.2) \quad \psi(r,s,t) = r+\lambda_1\left(\frac{s}{r}\right) + \lambda_2\left(\frac{2rs+t+s}{r^2+s}\right).$$

Since $\frac{f.g.h}{z^3} \neq 0$ for $z \in U$, from the definition of $M(\lambda_1,\lambda_2)$, we have

$$\frac{zf'}{f} = r \neq 0$$

and

$$\frac{zg'}{f} = r^2+s \neq 0, \quad \text{for} \quad z \in U.$$

Let $q(z) = \frac{1+z}{1-z}$ and

$B=B_1 \cup B_2$ where $B_1 = \{(r,s): r^2+s=0\}$ and $B_2 = \{(r,s): r=0\}$.

Then $\psi(r,s,t)$ is continuous in D where D is a component of $\{(C \times C \setminus B) \times C\}$ such that $(q(0),0,0) \in D$.

Let $\Omega = \{w \in C: \text{Re } w > 0\}$.

Since $(q(0),0,0) = 1 > 0$, $(q(0),0,0) \in \Omega$.

For $|\xi| = 1$, i.e. $\xi = e^{i\theta}(0 \leq \theta < 2\pi)$

$$q(\xi) = \frac{1+e^{i\theta}}{1-e^{i\theta}} = i \cot \frac{\theta}{2} = ir_1 \quad \text{(say)}, \tag{3.2.3}$$

and

$$\xi q'(\xi) = -\frac{1+r_1^2}{2} \quad \text{where } r_1 \text{ is as given above.} \tag{3.2.4}$$

Also

$$\left[1+\frac{zq''(z)}{q'(z)}\right] = \frac{1+z}{1-z}.$$

Therefore,

$$\text{Re}\left[1+\frac{\xi q''(\xi)}{q'(\xi)}\right] = \text{Re}\left[\frac{1+e^{i\theta}}{1-e^{i\theta}}\right] = 0.$$

Let $r_o = q(\xi)$, $s_o = m\xi q''(\xi)$ and

$$\text{Re}\left[1+\frac{t_o}{s_o}\right] = m \text{ Re}\left[1+\frac{\xi q''(\xi)}{q'(\xi)}\right] = 0. \tag{3.2.5}$$

Then in view of (3.2.3), (3.2.4) and (3.2.5), we have

$$\text{Re} \quad (r_o,s_o,t_o) = \lambda_2 \cdot \text{Re}\left[1+\frac{t_o}{s_o}\right] \leq 0$$

In view of Definition 3.2.2, $\psi \in \Psi(\Omega,q)$.

Theorem 3.2.1 provides that $p(z) \ll q(z)$ and therefore $f \in S^*$. This proves the first part of the theorem.

Let $\lambda_1+\lambda_2 \geq 1$. We have to show that

$$\mathrm{Re}\, \frac{zg'(z)}{g(z)} = \mathrm{Re}\, [1+ \frac{zf''(z)}{f'(z)}] > 0, \quad \text{for} \quad z \in U.$$

Let $\frac{zg'}{g} = P(z)$. Suppose there exists a $z_o \in U$ such that

(3.2.6) $\quad \mathrm{Re}\, P(z_o) = 0$ and $\mathrm{Re}\, P(z) \geq 0$, for $|z| \leq |z_o|$.

One can easily check that

$$\text{(3.2.7)} \qquad \mathrm{Re}\, \frac{z_o f'(z_o)}{f(z_o)} \geq 0$$

Also,

$$\frac{zh'(z)}{h(z)} = P(z) + \frac{zP'(z)}{P(z)}$$

$$= P(z) + \frac{\partial}{\partial\theta} \arg P(z).$$

Since $P(z) \geq 0$ for $|z| \leq |z_o|$ and $\mathrm{Re}\, P(z_o) = 0$, $\arg P(z)$ attains a local maximum or a local minimum at z_o.

Hence,

$$\text{(3.2.8)} \qquad \frac{\partial}{\partial\theta} \arg P(z_o) = 0$$

Therefore,

$$\mathrm{Re}\, \{(1-\lambda_1-\lambda_2) \frac{z_o f'(z_o)}{f(z_o)} + \lambda_1 \frac{z_o g'(z_o)}{g(z_o)} + \lambda_2 \frac{z_o h'(z_o)}{h(z_o)}\} \leq 0$$

follows in view of (3.2.6), (3.2.7) and (3.2.8), which is contradiction. Hence,

$$\operatorname{Re} P(z) > 0, \quad \text{for} \quad z \in U$$

and therefore $f(z) \in K$.

This completes the proof of the theorem.

Following corollaries can be easily deduced from Theorem 3.2.1.

<u>Corollary 3.2.1.</u> Let f belong to $M(\lambda_1,\lambda_2)$ and $\lambda_2 \leq 0$. Then for $0 \leq x \leq 1$,

$$M(\lambda_1,\lambda_2) \subseteq M(\lambda_1 x, \lambda_2 x) \subseteq S^*.$$

<u>Corollary 3.2.2.</u> Let $f \in M(\lambda_1,\lambda_2)$ and $\lambda_1 + \lambda_2 \geq 1$. Then for $0 \leq x \leq 1$,

$$M(\lambda_1, \lambda_2) \subseteq M(\lambda_1 x + (1-x), \lambda_2 x) \subseteq K.$$

3.3 In this section we prove the subordination of a certain class of functions having N initial missing coefficients to N-fold symmetric functions. The use of following lemmas is made in proving our main theorem.

<u>Lemma 3.3.1.</u> Let $p(z) = 1+p_N z^N + p_{N+1} z^{N+1} + \ldots$ and $q(z) = 1+b_1 z + b_2 z^2 + \ldots$ with $b_1 \neq 0$. If $p(z) \ll q(z)$, then $p(z) \underset{N}{\ll} q(z^{\cdot})$.

<u>Proof.</u> Since $p(z) \ll q(z)$,

$$p(z) = q(\omega(z)).$$

Also, $b_1 \neq 0$ provides that $\omega(z) = \sum_{k=N}^{\infty} c_k z^k$.

Therefore,

$$\omega(z) = z^N \varphi(z) \quad \text{where} \quad |\varphi(z)| < 1$$

Hence,

$$p(z) \underset{N}{<<} q(z).$$

<u>Lemma 3.3.2.</u> Let $\beta > 0$, $\beta+\gamma > 0$ and $\beta\alpha+\gamma \geq 0$ $(\alpha < 1)$. Then the differential equation

$$(3.3.1) \qquad q(z) + \frac{zq'(z)}{\beta q(z)+\gamma} = \frac{1+(1-2\alpha)z}{1-z}$$

has the univalent solution in U given by

$$(3.3.2) \qquad g(z) = \frac{z^{\beta+\gamma}(1-z)^{-2\beta(1-\alpha)}}{\beta\int_0^z t^{\beta+\gamma-1}(1-t)^{-2\beta(1-\alpha)}dt} - \frac{\gamma}{\beta}$$

<u>Proof.</u> More general form of this lemma is given in [57]

<u>Theorem 3.3.1.</u> Let $f(z) = z+\sum_{k=N+1}^{\infty} a_k z^k$ belong to $S^*(\alpha)$ $(\alpha < 1)$. If $\beta > 0$, $\beta+\gamma > 0$, $\beta\alpha+\gamma \geq 0$,

$$F(z) = \left(\frac{\beta+\gamma}{z^\gamma}\int_0^z f^\beta(w)w^{\gamma-1}dw\right)^{1/\beta}$$

and

$$K(z) = \left(\frac{\beta+\gamma}{z^\gamma}\int_0^z \frac{t^{\beta+\gamma-1}}{(1-t^N)^{\frac{2(1-\alpha)}{N}}}dt\right)^{1/\beta},$$

then

$$(3.3.3) \qquad \frac{zF'(z)}{F(z)} << \frac{zK'(z)}{K(z)} << \frac{1+(1-2\alpha)z^N}{1-z^N}.$$

<u>Proof</u>. Let

$$p(z) = \frac{zF'(z)}{F(z)} \quad \text{and} \quad q(z) = \frac{zK'(z)}{K(z)} .$$

Then

$$(3.3.4) \qquad p(z) + \frac{zp'(z)}{\beta p(z)+\gamma} = \frac{zf'(z)}{f(z)} \ll \frac{1+(1-2\alpha)z}{1-z}$$

and

$$(3.3.5) \qquad q(z) + \frac{zq'(z)}{\beta q(z)+\gamma} = \frac{1+(1-2\alpha)z^N}{1-z^N} \ll \frac{1+(1-2\alpha)z}{1-z}$$

(3.3.4) and (3.3.5) with the help of Lemma 2.2.1 provide

$$p(z) \ll \frac{1+(1-2\alpha)z}{1-z}$$

and

$$q(z) \ll \frac{1+(1-2\alpha)z}{1-z}$$

respectively.

Since $p(z)$ has initial gap and $q(z)$ has symmetrical gaps, Lemma 3.3.1 gives

$$p(z) \ll \frac{1+(1-2\alpha)z^N}{1-z^N}$$

and

$$q(z) \ll \frac{1+(1-2\alpha)z^N}{1-z^N} .$$

It only remains to show that $p(z) \ll q(z)$.

Since $q(z)$ has symmetrical gaps and $q(0) = 1$, it follows from (3.3.5) that there exists a function $h(z)$ analytic in U such that

$$(3.3.6) \qquad h(z) + \frac{Nzh'(z)}{\beta h(z)+\gamma} = \frac{1+(1-2\alpha)z}{1-z}$$

Also,

$$p(z) + \frac{zp'(z)}{\beta p(z)+\gamma} \ll \frac{1+(1-2\alpha)z}{1-z} = g(z) \qquad \text{(say)}.$$

Lemma 3.3.2 provides that $h(z)$ is univalent in U.

Let $p_\rho(z) = p(\rho z)$, $h_\rho(z) = h(\rho z)$ and $g_\rho(z) = g(\rho z)$.
Also, we can deduce from (3.3.6) and (3.3.4)

$$h_\rho(z) + \frac{Nzh'_\rho(z)}{\beta h_\rho(z)+\gamma} = g_\rho(z)$$

and

$$p_\rho(z) + \frac{Nzp'_\rho(z)}{\beta p_\rho(z)+\gamma} \ll g_\rho(z)$$

First, we will show that $p(z) \ll h(z)$.

Let

$$\psi(r,s) = r + \frac{s}{\beta r+\gamma}$$

$\psi(r,s)$ is holomorphic in $(C \times \{C \setminus \frac{\gamma}{\beta}\})$. Since $\psi(h_\rho(0),0) = g_\rho(0) = 1$, due to Theorem 3.2.4, it is sufficient to show that $\psi(h_\rho(\xi), m\xi h'_\rho(\xi)) \notin g_\rho(U)$ for $0 < \rho < 1$, $m \geq N$ and $|\xi| = 1$.

$$\psi(h_\rho(\xi), m\xi h'_\rho(\xi)) = h_\rho(\xi) + \frac{m\xi h'_\rho(\xi)}{\beta h_\rho(\xi)+\gamma}$$

$$= h_\rho(\xi) + \frac{m}{N}(g_\rho(\xi)-h_\rho(\xi)) \notin g_\rho(U)$$

follows in view of the fact that $m \geq N$.

Hence,

$$p_\rho(z) \ll h_\rho(z).$$

Letting $\rho \to 1$, we get

$$p(z) \ll h(z).$$

Using Lemma 3.3.1, we get

$$p(z) \ll h(z^N) = q(z).$$

This completes the proof of the theorem.

Corollary 3.3.1 56 Let $f \in S^*$ and

$$F(z) = \frac{2}{z} \int_0^z f(t)dt$$

If

$$K(z) = \frac{2}{z} \int_0^z \frac{t}{(1+t)^2}\, dt ,$$

then

$$\frac{zF'(z)}{F(z)} \ll \frac{zK'(z)}{K(z)} \ll \frac{1-z}{1+z} .$$

Corollary 3.3.2. Let $f(z) = z + \sum_{k=N+1}^{\infty} a_k z^k$ belong to $S_N^*(\alpha)$ $(-1 \leq \alpha < 1)$. If

$$F(z) = \frac{2}{z} \int_0^z f(t)dt$$

and

$$K(z) = \frac{2}{z} \int_0^z \frac{t}{(1-t^N)^{\frac{2(1-\alpha)}{N}}}\, dt ,$$

$$\frac{zF'(z)}{F(z)} \ll \frac{zK'(z)}{K(z)} \ll \frac{1+(1-2\alpha)z^N}{1-z^N} .$$

3.4 In this section we obtain the order of starlikness of $I_{\beta,\gamma}(f)$ for $f \in S_N^*(\alpha)$ $(\alpha < 1)$ (i.e. $f(z) \in S^*(\alpha)$ having N initial missing coefficients)

where

$$I_{\beta,\gamma}(f) = \left(\frac{\beta+\gamma}{z^{\gamma}} \int_0^z f^{\beta}(w) w^{\gamma-1} dw\right)^{1/\beta}.$$

For that we need the following lemma.

Lemma 3.4.1. Let $\mu(t)$ be a positive measure on $[0,1]$ and let $Q(z,t)$ be a complex valued function defined on $U \times [0,1]$ such that $Q(z,t)$ is integrable on $[0,1]$ for $z \in U$. Suppose that $\operatorname{Re} Q(z,t) > 0$ in U, $Q(re^{i\theta_0},t)$ $(0 \le \theta_0 \le 2\pi)$ is real and

$$\operatorname{Re} \frac{1}{Q(z,t)} \geq \frac{1}{Q(re^{i\theta_0},t)}$$

If

$$Q(z) = \int_0^1 Q(z,t)\, d\mu(t),$$

then

$$\operatorname{Re} \frac{1}{Q(z)} \geq \frac{1}{Q(re^{i\theta_0})} .$$

Proof. We will prove this lemma on the lines of proof of the lemma due to J. Feng and D.R. Wilken [14]

First, we will show that if $g(z)$ is analytic in U and $g(re^{i\theta_0}) > 0$ $(0 \le \theta_0 < 2\pi)$ is real, then the following statements are equivalent

(i) $\quad \operatorname{Re} \frac{1}{g(z)} \geq \frac{1}{g(re^{i\theta_o})}$ for $|z| \leq r$

(ii) $\quad \left| g(z) - \frac{g(re^{i\theta_o})}{2} \right| \leq \frac{g(re^{i\theta_o})}{2}$, for $|z| \leq r$

(ii) implies that

$$\left| g(z) - \frac{g(re^{i\theta_o})}{2} \right|^2 \leq \frac{(g(re^{i\theta_o}))^2}{4}$$

$$\Longleftrightarrow |g(z)|^2 - g(re^{i\theta_o}) \operatorname{Re} g(z) \leq 0$$

$$\Longleftrightarrow \operatorname{Re} \frac{1}{g(z)} \geq \frac{1}{g(re^{i\theta_o})} .$$

Since $\operatorname{Re} Q(z,t) > 0$ in U and $\mu(t)$ is a positive measure, $\operatorname{Re} Q(z) > 0$. Also, $Q(re^{i\theta_o},t)$ real. Hence, by the above argument

$$\left|Q(z) - \tfrac{1}{2} Q(re^{i\theta_o})\right| \leq \left|\int_0^1 Q(z,t)d\mu(t) - \tfrac{1}{2} \int_0^1 Q(re^{i\theta_o},t)d\mu(t)\right|$$

$$\leq \int_0^1 \left|Q(z,t) - \tfrac{1}{2} Q(re^{i\theta_o},t)\right| d\mu(t)$$

$$\leq \tfrac{1}{2} \int_0^1 Q(re^{i\theta_o},t)\, d\mu(t) = \tfrac{1}{2} Q(re^{i\theta_o}).$$

Again using the above argument we have

$$\operatorname{Re} \frac{1}{Q(z)} \geq \frac{1}{Q(re^{i\theta_o})}$$

This completes the proof of the lemma.

Theorem 3.4.1. Let $f(z) \in S_N^*(\alpha)$ $(\alpha < 1)$. If $\beta, \gamma \in R$ are such that $\beta > 0$, $\beta+\gamma > 0$, $\beta\alpha+\gamma \geq 0$ and

$$(3.4.1) \qquad g(z) = \left(\frac{\beta+\gamma}{z^{\gamma}} \int_0^z f^{\beta}(w) w^{\gamma-1} dw\right)^{1/\beta}$$

then the order of starlikness of $g(z)$ is given by

$$(3.4.2) \qquad \delta_N(\alpha,\beta,\gamma) = \operatorname*{Inf}_{|z|<1} \operatorname{Re} q(z)$$

where

$$(3.4.3) \qquad q(z) = \frac{z^{\beta+\gamma}(1-z^N)^{-\frac{2\beta(1-\alpha)}{N}}}{\beta \int_0^z t^{\beta+\gamma-1}(1-t^N)^{-2\beta(\frac{1-\alpha}{N})} dt} - \frac{\gamma}{\beta}$$

Moreover, if $\alpha \in [\alpha_0, 1)$ where

$$(3.4.4) \qquad \alpha_0 = \max \left\{ \frac{\beta-\gamma-N}{2\beta}, -\frac{\gamma}{\beta} \right\}$$

and g as given in (3.4.1), then

$$(3.4.5) \qquad \operatorname{Re} \frac{zg'(z)}{g(z)} = \frac{1}{\beta}\left[\frac{\beta+\gamma}{G\left(1, \frac{2\beta(1-\alpha)}{N}; \frac{\beta+\gamma}{N}+1; \frac{r^N}{1+r^N}\right)} - \gamma\right]$$

for $|z| \leq r < 1$.

and

$$(3.4.6) \qquad \delta_N(\alpha,\beta,\gamma) = \frac{1}{\beta}\left[\frac{\beta+\gamma}{G\left(1, \frac{2\beta(1-\alpha)}{N}; \frac{\beta+\gamma}{N}+1; \frac{1}{2}\right)} - \gamma\right]$$

where $G(a,b;c;z)$ is the hypergeometric function.

The extremal function is given by

$$g_0(z) = \left(\frac{\beta+\gamma}{z^{\gamma}}\int_0^z k(w)w^{\gamma-1}dw\right)^{1/\beta}$$

where $k(z) = \dfrac{z}{(1-z^N)^{\frac{2(1-\alpha)}{N}}}$

Proof. Theorem 3.3.1 provides

$$(3.4.7)\quad \mathrm{Re}\,\frac{zg'}{g} \geq \mathrm{Re}\,\frac{z^{\beta+\gamma}\,(1-z^N)^{-2\beta(\frac{1-\alpha}{N})}}{\beta\int_0^1 t^{\beta+\gamma-1}(1-t^N)^{-2\beta(\frac{1-\alpha}{N})}dt} - \frac{\gamma}{\beta}$$

(3.4.2) follows from (3.4.7).

Also, we can write R.H.S. of (3.4.7) as

$$\frac{1}{\beta}\,Q(z) - \frac{\gamma}{\beta}$$

where

$$Q(z) = \int_0^1 \left(\frac{1-z^N}{1-t^N z^N}\right)^{\frac{2\beta(1-\alpha)}{N}} t^{\beta+\gamma-1}\,dt$$

$$(3.4.8)\qquad = \frac{1}{N}\int_0^1 \left(\frac{1-z^N}{1-tz^N}\right)^{\frac{2\beta(1-\alpha)}{N}} t^{\beta+\gamma-1}\,dt.$$

Next we shall use the following well known formulae of hypergeometric functions [95]

$$(3.4.9)\quad \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}dt = \frac{\Gamma(b)\,\Gamma(c-b)}{\Gamma(c)}\,G(a,b;c,z),$$

$$c > b > 0$$

and

(3.4.10) $$G(a,b,c;z) = (1-z)^{-a}\, G(a, c-b; c; -\frac{z}{1-z}), \quad z \in \mathbb{C} \quad (1,\infty).$$

Let $\alpha \in (\alpha_o, 1)$ where α_o is given by (3.4.4) and $a = \frac{2\beta(1-\alpha)}{N}$, $b = \frac{\beta+\gamma}{N}$ and $c = \frac{\beta+\gamma}{N} + 1 = b+1$. Since $c>b>0$, by using (3.4.9) and (3.4.10), we get

$$Q(z) = \frac{(1-z^N)^a}{N} \frac{\Gamma(b)\,\Gamma(c-b)}{\Gamma(b)}\; G(a,b;c;z^N)$$

$$= \frac{1}{\beta+\gamma}\, G(a, c-b; c; -\frac{z^N}{1-z^N})$$

(3.4.11) $$= \frac{1}{\beta+\gamma}\, G(1, a; c; -\frac{z^N}{1-z^N})$$

Since $\alpha > \alpha_o$ implies $c > a$, the use of (3.4.9) in (3.4.11) provides

$$Q(z) = \int_0^1 \frac{1-z^N}{1-(1-t)z^N}\, d\mu(t), \quad \text{for} \quad z \in U$$

where

$$d\mu(t) = \frac{\Gamma(b)}{\Gamma(a)\,\Gamma(c-b)}\, t^{a-1}(1-t)^{c-a-1}dt.$$

If we let $Q(z,t) = \frac{1-z^N}{1-(1-t)z^N}$, then $\operatorname{Re} Q(z,t) > 0$, $Q(re^{\frac{\pi i}{N}}, t)$ is real

and

$$\operatorname{Re} \frac{1}{Q(z,t)} \geq \operatorname{Re} \frac{1-(1-t)z^N}{1-z^N} \geq \frac{1+(1-t)r^N}{1+r^N} = \frac{1}{Q(re^{\frac{\pi i}{N}}, t)}$$

for $|z| \leq r < 1$ and $0 \leq t \leq 1$.

Lemma 3.4.1 provides

$$\operatorname{Re} \frac{1}{Q(z)} \geq \frac{1}{Q(re^{\frac{i\pi}{N}})}$$

which implies (3.4.5). (3.4.6) can be obtained by letting $r \to 1^-$.

This completes the proof of the theorem.

Corollary 3.4.1. Let $f(z) = z+a_{N+1}z^{N+1}+\ldots$ belong to $M(\lambda)$, $\lambda \geq \frac{1}{N}$. Then the order of starlikeness of $f(z)$ is given by

$$\delta_N(0, \frac{1}{\lambda}, 0) = \frac{\Gamma(\frac{1}{N\lambda} + \frac{1}{2})}{\sqrt{\pi}\,\Gamma(\frac{1}{N\lambda} + 1)} \geq \frac{1}{2} .$$

As a special case we deduce that if $f(z) = z+a_{N+1}z^{N+1} + a_{2N+1}z^{2N+1}+\ldots\ldots$ belong to $M(\frac{1}{N})$, then $f \in S^*(\frac{1}{2})$.

Corollary 3.4.2. Let $\beta > 0$, $\beta+\gamma \geq N$. In this case α_0 given by (3.4.4) is equal to $\frac{\beta-\gamma-N}{2\beta}$. If we take $\alpha = \alpha_0$ i.e. $c=a$, we easily get

$$q(z) = \frac{\beta+\gamma z^N}{\beta(1-z^N)}$$

and

$$\delta_N(\frac{\beta-\gamma-N}{2\beta} ; \beta, \gamma) = \frac{\beta-\gamma}{2\beta}$$

which shows that, if $\beta \geq \gamma \geq N-\beta$, the integral operator $I_{\beta,\gamma}$ maps each starlike function of order $\frac{\beta-\gamma-N}{2\beta}$ with initial gap of width N to a starlike (univalent) function.

As a special case we deduce

The Libera integral operator maps starlike functions of order -1 with vanishing second coefficient to a starlike function.

Example 3.4.1. Let

$$f(z) = \frac{z}{(1-z^2)^2}$$

One can easily check that $f(z) \in S_2^*(-1)$.

Hence,

$$L(f) = \frac{2}{z} \int_0^z \frac{w}{(1-w^2)^2}\, dw = \frac{z}{1-z^2} = g(z) \quad \text{(say)}$$

and therefore,

$$\operatorname{Re} \frac{zg'(z)}{g(z)} = \operatorname{Re} \frac{1+z^2}{1-z^2} > 0 \;, \quad \text{for} \quad z \in U.$$

CHAPTER IV

ORDER OF STARLIKENESS OF CERTAIN INTEGRAL OPERATORS

4.1 In this chapter we first define certain subclasses of H.

Definition 4.1.1. A function $f(z)$ in H is said to belong to $Q(\alpha,A,B)$ $(\alpha \geq 0)$, $(-1 \leq B < A \leq 1)$ if $\frac{f(z)}{z} \cdot f'(z) \neq 0$ and

$$(4.1.1) \qquad \left\{(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1+\frac{zf''(z)}{f'(z)}\right)\right\} \ll \frac{1+Az}{1+Bz}, \text{ for } z \in U.$$

The class $Q(\alpha,A,B)$ is called Mocanu-Janowiski class defined by Jakubowski and Kaminski [31]. One can easily show that $Q(\alpha,A,B) \subset S^*(A,B)$.

Let $S(p)$ denote the class of functions $f(z)$ analytic in U and of the form

$$(4.1.2) \qquad f(z) = z^p + \sum_{m=p+1}^{\infty} a_m z^m$$

The subclasses $S_p^*(A,B)$, $K_p(A,B)$ and $Q_p(\alpha,A,B)$ of $S(p)$ are defined as follows:

Definition 4.1.2. A function $f(z)$ in $S(p)$ is said to belong to $S_p^*(A,B)$ $(-1 \leq B < A \leq 1)$ if

$$\frac{1}{p}\,\frac{zf'(z)}{f(z)} \ll \frac{1+Az}{1+Bz}, \quad \text{for } z \in U.$$

$S_p^*(1-2\alpha,-1) = S_p^*(\alpha)$ is well known subclass of $S(p)$ of p-valent starlike functions of order α.

Definition 4.1.3. A function $f(z)$ in $S(p)$ is said to belong to $K_p(A,B)$ $(-1 \leq B < A \leq 1)$ if

$$(4.1.4) \qquad \frac{1}{p}\left(1+\frac{zf''(z)}{f'(z)}\right) \ll \frac{1+Az}{1+Bz}, \quad \text{for } z \in U.$$

One can easily see that $f(z)$ belongs to $K_p(A,B)$ if and only if $zf'(z)$ belongs to $S_p^*(A,B)$. $K_p(1-2\alpha,-1) = K_p(\alpha)$ is well known subclass of $S(p)$ of p-valent convex functions of order α.

Definition 4.1.4. A function $f(z)$ in $S(p)$ is said to belong to $Q_p(\alpha,A,B)$ $(\alpha \geq 0)$, $(-1 \leq B < A \leq 1)$ if $f(z).f'(z) \neq 0$ in $0 < |z| < 1$ and

$$\frac{1}{p}\left\{(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1+\frac{zf''(z)}{f'(z)}\right)\right\} \ll \frac{1+Az}{1+Bz}, \quad \text{for } z \in U.$$

The class $Q_p(\alpha,A,B)$ is analog of the class $Q(\alpha,A,B)$ defined by Jakubowski and Kaminski [31] . One can easily show that $Q_p(\alpha,A,B) \subseteq S_p^*(A,B)$ and for $\alpha \geq p$, $Q_p(\alpha,A,B) \subseteq K_p(A,B)$.

The subclass $S^*(A,B)$ $(-1 \leq B < A \leq 1)$ of S^* has been defined in Chapter I. We continue to call a function to be in $S^*(A,B)$ if $A,B \in R$, $|B| \leq 1$, $A \neq B$ and

$$\frac{zf'(z)}{f(z)} \ll \frac{1+Az}{1+Bz}, \quad \text{for } z \in U.$$

Although it may not be starlike.

Similarly we continue to call a function $f(z)$ belonging to $S(p)$ to be in $S_p^*(A,B)$ if

$$\frac{1}{p}\,\frac{zf'(z)}{f(z)} \ll \frac{1+Az}{1+Bz}$$

where A and $B \in R$, $|B| \leq 1$, $A \neq B$.

Let $\beta, \gamma \in R$, $\beta > 0$ and $\beta+\gamma > 0$, consider the integral operator $I_{\beta,\gamma}(f) = g$, $f \in S$ defined by

$$(4.1.6) \qquad g(z) = \left(\frac{\beta+\gamma}{z^\gamma}\int_0^z f^\beta(w)w^{\gamma-1}dw\right)^{1/\beta}, \quad z \in U.$$

If A and B are such that $A \neq B$, $|B| \leq 1$ and $|\beta A + \gamma B| \leq \beta+\gamma$, then from a more general result on Briot-Bouquet differential subordination [57] it is easy to deduce that the integral operator $I_{\beta,\gamma}(f)$ maps $S^*(A,B)$ into $S^*(A,B)$. We define the order of starlikeness of the class $I_{\beta,\gamma}(S^*(A,B))$ by the largest number $\delta=\delta(\beta,\gamma;A,B)$ such that

$$(4.1.7) \qquad I_{\beta,\gamma}(S^*(A,B)) \subset S^*(\delta).$$

P.T. Mocanu, D. Ripeanu and I. Serb in [63] found the order of starlikeness of $I_{\beta,\gamma}(S^*(\alpha))$ $(\alpha < 1)$. We extend the results of Mocanu, Ripeanu and Serb [63] to $I_{\beta,\gamma}(S^*(A,B))$ where $A,B \in R$, $A \neq B$, $|B| \leq 1$ and $|\beta A+\gamma B| \leq \beta+\gamma$. As corollaries, we obtain the order of starlikeness of $Q(\alpha,A,B)$ and $L(S^*(A,B))$ where L is the

Libera transform. Our general result includes some particular ones obtained by several authors [59, 61, 62, 63]

Let us define the operator $I^p_{\beta,\gamma}(f) = h(z)$, $f \in S(p)$ by

$$(4.1.8) \qquad h(z) = \left(\frac{\beta p+\gamma}{z^\gamma}\int_0^z f^\beta(w)w^{\gamma-1}dw\right)^{1/\beta}$$

K.S. Padmanabhan and G. Lakshma Reddy have shown that if $g(z) = \frac{p+c}{z^c}\int_0^z t^{c-1}f(t)dt$, $c=1,2,3\ldots$, then $g(z)$ belongs to $S^*_p(A,B)$ whenever $f(z)$ belongs to $S^*_p(A,B)$ where $(-1 \leq B < A \leq 1)$. With the help of Briot-Bouquet differential subordination [57], one can easily show that if $\beta > 0$, $\gamma > 0$, $-1 \leq B < A \leq 1$, then $h(z)$ belongs to $S^*_p(A,B)$ whenever $f(z)$ belongs to $S^*_p(A,B)$. This extends the results of K.S. Padmanabhan and G. Lakshma Reddy [69]. We find the order of p-valent starlikeness of $I^p_{\beta,\gamma}(S^*(A,B))$. As a corollary, we show that if $f(z)$ belongs to $S^*_p(0)$ and $g(z) = \left(\frac{2p-1}{z^p}\int_0^z f(w)w^{p-2}dw\right)$, then $g(z)$ belongs to $S^*_p(\frac{1}{2p})$. Further we show that if $f(z) \in S(p)$ and $\left|\frac{zf''(z)}{f'(z)} - (p-1)\right| < \lambda p$, then $f(z)$ belongs to $S^*_p(\alpha_0)$ where $\alpha_0 = (p\int_0^1 t^{p-1}e^{\lambda(1-t)^{-1}}dt)$.

4.2 We will use the following lemmas to prove our main theorem.

<u>Lemma 4.2.1.</u> Let $\beta > 0$, $\beta+\gamma > 0$. If A and B are such that

$A, B \in R$, $A \neq B$, $|\beta| \leq 1$ and $|\beta A + \gamma B| \leq \beta + \gamma$, then the differential equation

$$(4.2.1) \quad q(z) + \frac{zq'(z)}{\beta q(z)+\gamma} = \frac{1+Az}{1+Bz}$$

has the univalent solution in U given by

$$(4.2.2) \quad q(z) = \frac{1}{\beta Q(z)} - \frac{\gamma}{\beta}$$

where

$$(4.2.3)\, Q(z) = \begin{cases} \displaystyle\int_0^1 \left(\frac{1+Btz}{1+Bz}\right)^{\beta\left(\frac{A-B}{B}\right)} t^{\beta+\gamma-1}\, dt, & \text{if } B \neq 0 \\ \displaystyle\int_0^1 e^{\beta A(t-1)z}\, t^{\beta+\gamma-1}\, dt, & \text{if } B = 0. \end{cases}$$

If $p(z) = 1+p_1 z + p_2 z^2 + \ldots$ is regular in U and satisfies the differential subordination

$$(4.2.4) \quad p(z) + \frac{zp'(z)}{\beta p(z)+\gamma} \ll \frac{1+Az}{1+Bz}, \quad \text{for } z \in U,$$

then $p(z) \ll q(z)$ and $q(z)$ will be the best dominant of (4.2.4). More general form of this lemma can be found in[57]

Lemma 4.2.2. Let $\mu(t)$ be a positive measure on $[0,1]$ and let $Q(z,t)$ be a complex valued function defined on $U \times [0,1]$ such that $Q(z,t)$ is integrable on $[0,1]$ and $\mathrm{Re}\, Q(z,t) > 0$ for $z \in U$.

Suppose

$$Q(z) = \int_0^1 Q(z,t)\,d\mu(t).$$

If $Q(-r,t)$ is real and

$$\operatorname{Re}\frac{1}{Q(z,t)} \geq \frac{1}{Q(-r,t)} \quad \text{for} \quad |z| \leq r < 1 \quad \text{and} \quad t \in [0,1]$$

then

(4.2.5) $$\operatorname{Re}\frac{1}{Q(z)} \geq \frac{1}{Q(-r)} \qquad \text{for} \quad |z| \leq r.$$

If $Q(r,t)$ is real and

$$\operatorname{Re}\frac{1}{Q(z,t)} \geq \frac{1}{Q(r,t)} \quad \text{for } |z| \leq r < 1 \text{ and } t \in [0,1] \;,$$

then

(4.2.6) $$\operatorname{Re}\frac{1}{Q(z)} \geq \frac{1}{Q(r)} \quad \text{for} \quad |z| \leq r.$$

Proof. First part of the lemma is due to J Feng and D.R. Wilken [14]. Second part follows from Lemma 3.4.1.

Now we will prove our main theorem.

Theorem 4.2.1. Let $\beta > 0$, $\beta+\gamma > 0$. Consider the integral operator

(4.2.7) $$g(z) = I_{\beta,\gamma}(f) = \left(\frac{\beta+\gamma}{z^{\gamma}}\int_0^z f^{\beta}(w)w^{\gamma-1}dw\right)^{1/\beta}$$

If $A,B \in R$, $A \neq B$, $|B| \leq 1$ and $|\beta A+\gamma B| \leq \beta+\gamma$, then the order of starlikeness of $I_{\beta,\gamma}(S^*(A,B))$ is given by

(4.2.8) $$\delta(\alpha,\beta;A,B) = \inf_{|z|<1} \operatorname{Re} q(z)$$

where $q(z)$ is given by (4.2.2).

Moreover, if B=0, then

$$\operatorname{Re}_{|z|\leq r<1} \frac{zg'(z)}{g(z)} \geq \frac{1}{\beta \int_0^1 t^{\beta+\gamma-1} e^{\beta|A|(1-t)r} dt} - \frac{\gamma}{\beta}$$

and

$$(4.2.9) \quad \delta(\beta,\gamma;A,0) \geq \frac{1}{\beta} \left(\int_0^1 t^{\beta+\gamma-1} e^{\beta|A|(1-t)} dt\right)^{-1} - \frac{\gamma}{\beta}$$

If $B < 0$ and $B\gamma+B+\beta A < 0$, then

$$\operatorname{Re}_{|z|\leq r<1} \frac{zg'(z)}{g(z)} \geq q(-r) = \frac{1}{\beta}\left[\frac{\beta+\gamma}{G(1,\frac{\beta(B-A)}{B};\beta+\gamma+1;\frac{Br}{Br-1})} - \gamma\right]$$

and

$$(4.2.10) \quad \delta(\beta,\gamma;A,B) = q(-1) = \frac{1}{\beta}\left[\frac{\beta+\gamma}{G(1,\frac{\beta(B-A)}{B};\beta+\gamma+1;\frac{B}{B-1})} - \gamma\right]$$

If $B > 0$ and $B\gamma+B+\beta A > 0$, then

$$\operatorname{Re}_{|z|\leq r<1} \frac{zg'(z)}{g(z)} \geq q(r) = \frac{1}{\beta}\left[\frac{\beta+\gamma}{G(1,\frac{\beta(B-A)}{B};\beta+\gamma+1;\frac{Br}{Br+1})} - \gamma\right]$$

and

$$(4.2.11) \quad \delta(\beta,\gamma;A,B) = q(1) = \frac{1}{\beta}\left[\frac{\beta+\gamma}{G(1,\frac{\beta(B-A)}{B};\beta+\gamma+1;\frac{B}{B+1})} - \gamma\right]$$

where $G(a,b;c;x)$ is the hypergeometric function. The extremal function is given by

$$g(z) = I_{\beta,\gamma}(k(z))$$

where

$$k(z) = \begin{cases} \dfrac{z}{(1+Bz)^{\frac{B-A}{B}}}, & \text{for } B \neq 0 \\ ze^{Az}, & \text{for } B = 0. \end{cases}$$

Proof. Let

$$p(z) = \frac{zg'(z)}{g(z)}$$

From (4.2.7), we obtain

$$p(z) + \frac{zp'(z)}{\beta p(z)+\gamma} = \frac{zf'(z)}{f(z)} .$$

Since $f(z) \in S^*(A,B)$, $p(z)$ satisfies the differential subordination (4 .2.4) and hence, $p(z) \ll q(z)$, $q(z)$ given by (4.2.2) which implies (4.2.8).

Let $B=0$. Then

$$q(z) = \frac{1}{\beta \int_0^1 e^{\beta A(t-1)z}\, t^{\beta+\gamma-1}\, dt} - \frac{\gamma}{\beta}$$

Using Lemma 4.2.2, we get (4.2.9). For proving (4.2.10) and (4.2.11), we shall need the following well known formulae of hypergeometric functions [95]. If $c > b > 0$, then

$$(4.2.12) \qquad \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tx)^{-a}dt = \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)} G(a,b;c;x)$$

and

$$(4.2.13) \qquad G(a,b;c;x) = (1-x)^{-a}\, G(a,c-b;c;\frac{x}{x-1})$$

for all $x \in C \ (1,\infty)$.

If we set $a = \frac{\beta(B-A)}{B}$, $b = \beta+\gamma$, $c = \beta+\gamma+1$, then $c > b > 0$ and $Q(z)$ given by (4.2.3) is

$$Q(z) = (1+Bz)^{a} \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)} G(a,b;c;-Bz)$$

$$= \frac{1}{\beta+\gamma} G(a,c-b;c;\frac{Bz}{1+Bz}) \tag{4.2.14}$$

If $B < 0$ and $B\gamma+B+\beta A < 0$, then $c > 0$. Again using (4.2.12) in (4.2.14) we get

$$Q(z) = \int_0^z \frac{1+Bz}{1+(1-t)Bz} d\mu(t)$$

where

$$d\mu(t) = \frac{\Gamma(b)}{\Gamma(c)\Gamma(c-b)} t^{a-1}(1-t)^{c-a-1}dt.$$

Let

$$Q(z,t) = \frac{1+Bz}{1+(1-t)Bz}.$$

Then $\operatorname{Re} Q(z,t) > 0$ for $|z| < 1$ and $t \in [0,1]$ and $Q(-r,t)$ is real. Since $\beta < 0$, therefore,

$$\operatorname{Re} \frac{1}{Q(z,t)} = \operatorname{Re} \left\{\frac{1+(1-t)Bz}{1+Bz}\right\} \geq \frac{1-(1-t)Br}{1-Br}$$

$$= \frac{1}{Q(-r,t)} \quad \text{for} \quad |z| \leq r < 1.$$

which provides (4.2.20).

We can prove (4.2.11) on similar lines using part of the Lemma 4.2.2.

This completes the proof of the theorem.

Corollary 4.2.1 [63] Let $\beta > 0$, $\beta+\gamma > 0$ and consider the integral operator $I_{\beta,\gamma}$ defined by (4.1.6). If $\alpha \in [-\frac{\gamma}{\beta}, 1)$ then the order of starlikeness of the class $I_{\beta,\gamma}(S^*(\alpha))$ is given by

$$\delta(\alpha;\beta,\gamma) = \operatorname*{Inf\,Re}_{|z|<1} q(z)$$

where $$q(z) = \left(\beta \int_0^1 \left(\frac{1-z}{1-tz}\right)^{2\beta(1-\alpha)} t^{\beta+\gamma-1} dt\right)^{-1} - \frac{\gamma}{\beta} .$$

Moreover if $\alpha \in [\alpha_0, 1)$, where

$$\alpha_0 = \max\left\{\frac{\beta-\gamma-1}{2\beta}, -\frac{\gamma}{\beta}\right\}$$

and $g = I_{\beta,\gamma}(f)$ for $f \in S^*(\alpha)$, then

$$\operatorname{Re}\frac{zg'(z)}{g(z)} \geq q(-r) = \frac{1}{\beta}\left[\frac{\beta+\gamma}{G(1, 2\beta(1-\alpha); \beta+\gamma+1; \frac{r}{1+r})} - \gamma\right]$$

for $|z| \leq r < 1$ and

$$\delta(\alpha;\beta,\gamma) = q(-1) = \frac{1}{\beta}\left[\frac{\beta+\gamma}{G(1, 2\beta(1-\alpha); \beta+\gamma+1; 1/2} - \gamma\right]$$

where G is the hypergeometric function. The extremal function is given by $g = I_{\beta,\gamma}(k)$, where

$$k(z) = \frac{z}{(1-\alpha)^{2(1-\alpha)}} .$$

Corollary 4.2.2 Let $f(z) \in S^*(A,B)$ $(-1 \leq B < A \leq 1)$ and

$$L(f) = \frac{2}{z}\int_0^z f(w)dw .$$

Then the order of starlikeness of $L(f)$ is given by

$$\delta(1,1,A,B) = \operatorname*{Inf}_{|z|<1} \operatorname{Re} q(z)$$

where

$$q(z) = \begin{cases} \left(\int_0^1 \left(\frac{1+Btz}{1+Bz}\right)^{\frac{A-B}{B}} t\, dt\right)^{-1} - 1, & \text{if } B \neq 0 \\ \left(\int_0^1 e^{A(t-1)z} t\, dt\right)^{-1} - 1, & \text{if } B = 0. \end{cases}$$

If $B > 0$, then the order of starlikeness of $L(f)$ is given by

$$\delta(1,1,A,B) = \left\{ \frac{2}{G(1, \frac{B-A}{B}; 3; \frac{B}{B+1})} \right\} - 1$$

If $B < 0$ and $2B+A$, then the order of starlikeness of $L(f)$ is given by

$$\delta(1,1,A,B) = \left\{ \frac{2}{G(1, \frac{B-A}{B}; 3; \frac{B}{B-1})} \right\} - 1$$

If $B = 0$, then the order of starlikeness of $f(z)$ is given by

$$\delta(1,1,A,0) = \left(\frac{A^2}{e^A - (A+1)} - 1 \right)$$

<u>Corollary 4.2.3.</u> Let $f(z)$ belong to $Q(\alpha,A,B)$ $(-1 \leq B < A \leq 1)$, $(\alpha > 0)$. The order of starlikeness of $f(z)$ is given by

$$\delta(\tfrac{1}{\alpha}, 0; A,B) = \operatorname*{Inf}_{|z|<1} \operatorname{Re} q(z)$$

where

$$q(z) = \begin{cases} \alpha\left(\int_0^1 \left(\frac{1+Btz}{1+Bz}\right)^{\frac{A-B}{B\alpha}} t^{1/\alpha-1}\, dt\right)^{-1}, & \text{if } B \neq 0 \\ \alpha\left(\int_0^1 e^{A\left(\frac{t-1}{\alpha}\right)z} t^{1/\alpha-1}\, dt\right)^{-1}, & \text{if } B = 0. \end{cases}$$

If $B > 0$, then the order of starlikeness of $f(z)$ is given by

$$\delta\left(\frac{1}{\alpha},0;A,B\right) = \frac{1}{G\left(1, \frac{B-A}{B\alpha}; 1/\alpha+1; \frac{B}{B+1}\right)}$$

If $B < 0$ and $B+\alpha^{-1}A < 0$, then the order of starlikeness of $f(z)$ is given by

$$\delta\left(\frac{1}{\alpha},0;A,B\right) = \frac{1}{G\left(1,\frac{B-A}{B\alpha}; 1/\alpha+1; \frac{B}{B-1}\right)} .$$

If $B=0$, then the order of starlikeness of $f(z)$ is given by

$$\delta\left(\frac{1}{\alpha},0;A,B\right) = \alpha\left(\int_0^1 e^{A\left(\frac{1-t}{\alpha}\right)} t^{1/\alpha-1}dt\right)^{-1}$$

All the above corollaries follow from Theorem 4.2.1.

Remark. Putting $\alpha=1$ in corollary 4.2.3 we obtain the order of starlikeness of $f(z) \in K(A,B)$. Also, we can deduce from Corollary 4.2.3 the following.

If $f(z) \in S$ and $\left|\frac{zf''(z)}{f'(z)}\right| < \lambda$, $(0 < \lambda < 1)$, then the order of starlikeness of $f(z)$ is given by

$$\delta(\lambda) = \frac{\lambda}{e^{\lambda}-1}$$

Corollary 4.2.4. Let $\beta > 0$ and $A,B \in R$, $A \neq B$, $0 < |B| \leq 1$. If

$$I_{\beta,\gamma}(f) = \left(\frac{\beta+\gamma}{z^{\gamma}}\int_{0}^{z} f^{\beta}(w)w^{\gamma-1}dw\right)^{1/\beta},$$

$\beta+\gamma \geq |B|$ and $\beta A + \gamma B + B = 0$, then $I_{\beta,\gamma}(f)$ maps $S^*(A,B)$ into $S^*(-\frac{B\gamma}{\beta}, B)$.

Proof. Since $\beta A+\gamma B+B=0$, $\frac{\beta(B-A)}{B} = \beta+\gamma+1$. Also $\beta+\gamma \geq |B|$ with $\beta A+\gamma B+B = 0$ implies $\beta+\gamma \geq |\beta A+\gamma B|$. Theorem 4.2.1 gives

$$Q(z) = \frac{1}{\beta+\gamma} G\left(1, \frac{\beta(B-A)}{B}; \beta+\gamma+1; \frac{Bz}{1+Bz}\right)$$

$$= \frac{1}{\beta+\gamma} G\left(1, \beta+\gamma+1; \beta+\gamma+1; \frac{Bz}{1+Bz}\right)$$

$$= \frac{1+Bz}{\beta+\gamma}$$

Therefore,

$$q(z) = \frac{1}{\beta}\frac{\beta+\gamma}{1+Bz} - \frac{\gamma}{\beta} = \frac{\beta-\gamma Bz}{\beta(1+Bz)}.$$

This completes the proof of the theorem.

Remark 4.2.1. If $\beta=1$, $\gamma=1$, $A=2$ and $B=-1$, then the conditions of the above corollary are satisfied and therefore, we can deduce that Libera integral operator maps starlike functions of order $-\frac{1}{2}$ into starlike functions which is due to Mocanu, Ripeanu and Serb [63]

Particular Cases 1. If we take $\gamma=0$, $\beta = 1/\alpha \leq 1$, $A = 1$ and $B = -1$, then

$$\delta(1/\alpha, 0; 1, -1) = \frac{1}{G(1, \frac{2}{\alpha}; \frac{1}{\alpha}+1; \frac{1}{2})}.$$

Using the well known formula of hypergeometric functions

$$G(a,b;\ \frac{a+b+1}{2}\ ;\ \frac{1}{2}) = \frac{\pi^{1/2}\Gamma(\frac{a+b+1}{2})}{\Gamma(\frac{a+1}{2})\Gamma(\frac{b+1}{2})},$$

we get

$$\delta(1/\alpha,0;1,-1) = \frac{\Gamma(\frac{1}{\alpha}+\frac{1}{2})}{\pi^{1/2}\Gamma(1/\alpha+1)}.$$

which is order of starlikeness of α-convex functions for $\alpha\geq 1$. This was obtained in [59]

2. If we take $\gamma=0$, $\beta=2$, $A>0$, $B=0$. We obtain the order of starlikeness of $Q(\frac{1}{2},A,0)$

$$\delta(2,0;A,0) = \frac{2A^2}{e^{2A}-(1+2A)}.$$

3. If we take $\beta=1$, $=0$, $A=1-2\alpha$ and $B=-1$

$$\delta(1,0;1-2\alpha,-1) = \frac{1}{G(1,2(1-\alpha);2,\frac{1}{2})}.$$

By expanding $G(1,2(1-\alpha);2,\ \frac{1}{2})$ in power series we obtain the following

$$\delta(1,0;1-2\alpha,-1) = \begin{cases} \dfrac{1-2(1-\alpha)}{2-2^{2(1-\alpha)}}, & \alpha\neq\frac{1}{2} \\[2ex] \dfrac{1}{2\log 2}, & \alpha=\frac{1}{2}. \end{cases}$$

Thus we get the order of starlikeness of convex functions of order α. Which is due to Zmorovich and Korobkova [99]

4. The order of starlikeness of $L(S^*(\frac{1}{2}))$ where L is the Libera integral operator defined by

$$L[f(z)] = \frac{2}{z} \int_0^z f(w)dw$$

is given by

$$\delta(1,1;0,-1) = \frac{2}{G(1,1;3,\frac{1}{2})} - 1$$

$$= \frac{1}{2(1-\ln 2)} - 1$$

This was obtained in [61]

4.3 In this section we will discuss about the order of p-valent starlikeness of $I^p_{\beta,\gamma}(S^*_p(A,B))$ where $I^p_{\beta,\gamma}(f)$ is defined by (4.1.8).

Theorem 4.3.1. Let $\beta > 0$, $\beta p + \gamma > 0$. Consider the integral operator

$$I^p_{\beta,\gamma}(f) = \left(\frac{\beta p+\gamma}{z^\gamma} \int_0^z f^\beta(w) w^{\gamma-1}\, dw\right)^{1/\beta} .$$

If $A,B \in R$, $A \neq B$, $|B| \leq 1$ and $|\beta pA+\gamma B| \leq \beta p+\gamma$ then the order of p-valent starlikeness of $I^p_{\beta,\gamma}(S^*_p(A,B)$ is given by

$$\delta(\beta p,\gamma;A,B) = \operatorname*{Inf}_{|z|<1} \operatorname{Re} q(z)$$

where

$$q(z)=\begin{cases} \dfrac{1}{\beta p \int_0^1 \left(\frac{1+Btz}{1+Bz}\right)^{\beta p\left(\frac{A-B}{B}\right)} t^{\beta+\gamma-1}\,dt} - \dfrac{\gamma}{\beta p}, & \text{if } B \neq 0 \\ \dfrac{1}{\beta p \int_0^1 e^{\beta p A(t-1)z}\, t^{\beta p+\gamma-1}\,dt} - \dfrac{\gamma}{\beta p}, & \text{if } B = 0 \end{cases}$$

If $B = 0$, then

$$\delta(\beta p,\gamma;A,B) = \frac{1}{\beta p \int_0^1 e^{\beta p|A|(1-t)}\, t^{\beta p+\gamma-1}} - \frac{\gamma}{\beta p}\,.$$

If $B < 0$ and $B\gamma+B+\beta Ap < 0$, then

$$\delta(\beta p,\gamma;A,B) = \frac{1}{\beta p}\,\frac{\beta p+\gamma}{G(1,\frac{(B-A)}{B}\beta p;\beta p+\gamma+1;\frac{B}{B-1})} - \frac{\gamma}{\beta p}$$

If $B > 0$ and $B\gamma+B+\beta Ap > 0$, then

$$\delta(\beta p,\gamma;A,B) = \frac{1}{\beta p}\,\frac{\beta p+\gamma}{G(1,\frac{(B-A)}{B}\beta p;\ \beta p+\gamma+1;\frac{B}{B+1})} - \frac{\gamma}{\beta p}$$

where $G(a,b;c;z)$ is hypergeometric function.

<u>Proof.</u> Let

$$\frac{1}{p}\,\frac{zf'(z)}{f(z)} = p(z).$$

Then (4.3.1) provides

$$p(z) + \frac{zp'(z)}{p\beta p(z)+\gamma} = \frac{1}{p}\ \frac{zf'(z)}{f(z)} \prec\prec \frac{1+Az}{1+Bz}\,.$$

The theorem follows directly from Theorem 4.2.1.

The following corollaries can be easily deduced from the above theorem.

Corollary 4.3.1. Let $\beta > 0$, $\beta p+\gamma>0$. Consider the integral operator $I^p_{\beta,\gamma}$ defined by (4.3.1). If $\alpha \in [-\frac{\gamma}{\beta p}, 1)$, then the order of p-valent starlikeness of $I^p_{\beta,\gamma}(S^*_p(\alpha))$ is given by

$$\delta(\alpha;\beta p,\gamma) = \operatorname{Inf}_{|z|<1} \operatorname{Re} \; q(z)$$

where

$$q(z) = \frac{1}{\beta p} \frac{}{\int_0^1 \left(\frac{1-z}{1-tz}\right)^{2\beta p(1-\alpha)} t^{\beta p+\gamma-1}\, dt} - \frac{\gamma}{\beta p}.$$

Moreover if $\alpha \in [\alpha_0, 1)$, where

$$\alpha_0 = \max\left\{\frac{\beta p-\gamma-1}{2\beta p} - \frac{\gamma}{\beta p}\right\},$$

then

$$\delta(\alpha;\beta p,\gamma) = \frac{1}{\beta p}\left[\frac{\beta p}{G(1, 2\beta p(1-\alpha); \beta p+\gamma+1; \frac{1}{2})} - \gamma\right]$$

$G(a,b;c;z)$ is the hypergeometric function. The extremal function is given by $g = I^p_{\beta,\gamma}(k)$, where

$$k(z) = \frac{z^p}{(1-z)^{2p(1-\alpha)}}.$$

Corollary 4.3.2. Let $f(z)$ belong to $S^*_p(0)$, $\beta=1$, $\gamma=p-1$, then

$$h(z) = \frac{2p-1}{z^{p-1}} \int_0^z f(w) w^{p-2}\, dw$$

belongs to $S_p^*(\frac{1}{2p})$. For $p=1$, it reduces to the well known result that every univalent convex function is starlike of order $\frac{1}{2}$.

Corollary 4.3.3. Let $f(z)$ belong to $S(p)$ and

$$\frac{1}{p}\left|\frac{zf''(z)}{f'(z)}-(p-1)\right| < \lambda \qquad 0 \leq \lambda < 1$$

then $f(z)$ is starlike of order $\left(p\int_0^1 t^{p-1}\, e^{\lambda(1-t)}dt\right)^{-1}$. The result is sharp for the function $f_0(z) = \int_0^z t^{p-1} e^{\lambda p t} dt$

Corollary 4.3.4. Let $f(z)$ belongs to $S(p)$. If $f(z).f'(z) \neq 0$ in $0 < |z| < 1$ and

$$\frac{1}{p}\left\{(1-\alpha)\frac{zf'}{f} + \alpha\left(1+\frac{zf''(z)}{f'(z)}\right)\right\} > \lambda \quad , \lambda \geq p.$$

Then the order of p-valent starlikeness is given by

$$\delta(\tfrac{p}{\lambda},0;1-2\alpha,1) = \frac{\Gamma(\frac{p}{\alpha}+\frac{1}{2})}{\pi^{1/2}\Gamma(\frac{p}{\alpha}+1)}.$$

CHAPTER V

CERTAIN SUBCLASSES OF UNIVALENT FUNCTIONS WITH INITIAL MISSING COEFFICIENTS

5.1 In this chapter we study certain subclasses of univalent functions with initial missing coefficients.

Definition 5.1.1. Let N be a fixed positive integer and H_N denote the class of functions $f(z) = z + \sum_{k=N+1}^{\infty} a_k z^k$ regular in U.

We define the following subclasses of H_N.

Definition 5.1.2. A function $f(z)$ in H_N is said to belong to $S_N^*(A,B)(-1 \leq B < A \leq 1)$ if it satisfies the following

$$(5.1.1) \qquad \frac{zf'(z)}{f(z)} \ll \frac{1+Az}{1+Bz}, \quad \text{for } z \in U.$$

Definition 5.1.3. A function $f(z)$ in H_N is said to belong $Q_N(\alpha,A,B)$ $(\alpha \geq 0)$, $(-1 \leq B < A \leq 1)$ if $\frac{f(z).f'(z)}{z} \neq 0$ and

$$(5.1.2) \qquad \{(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha(1+\frac{zf''(z)}{f'(z)})\} \ll \frac{1+Az}{1+Bz}, \text{ for } z \in U.$$

Definition 5.1.4. A function $f(z)$ in H_N is said to belong to $R_N^{\alpha}(A,B)$ if

$$(5.1.3) \qquad f'(z) \ll \frac{1+(A\cos\alpha - iB\sin\alpha)e^{i\alpha}z}{1+Bz}$$

where $(-1 \leq B < A \leq 1)$ and $(|\alpha| < \frac{\pi}{2})$.

One can easily see that $S_N^*(A,B)$ and $Q_N(\alpha,A,B)$ being the subclasses of $S^*(A,B)$ and $Q(\alpha,A,B)$ respectively, consist of univalent functions only. If $f(z)$ belongs to $R_N^\alpha(A,B)$, then $\mathrm{Re}\, e^{i\alpha}f(z) > 0$ for $z \in U$, by famous Noshiro-Warachawski theorem $R_N^\alpha(A,B)$ also consists of univalent functions.

Coefficient estimates, distortion properties and a coefficient inequality have been established for $f(z)$ in the class $S_N^*(A,B)$ and $R_N^\alpha(A,B)$. The sharp estimates for arc length and area of $f(z) \in R_N^\alpha(A,B)$ have been obtained. A condition depending upon λ, A, B and α have been deduced such that $(1-\lambda)\, z+\lambda(f(z) * zh'(z))$ belongs to $R_N^\alpha(A,B)$ whenever $f(z)$ and $h(z)$ belong to $R_N^\alpha(A,B)$. The integral representation, distortion properties and some coefficient estimates have been established for $f(z)$ belonging to $Q_N(\alpha,A,B)$. It has been observed that the extremal functions for coefficient and distortion for the above classes are N-fold symmetric functions. The results of this chapter yield along with some other results, results obtained by Goel and Mehrok [16], Jakubowiski Z.J. and Kaminski J. [31], Juneja and Mogra [35], Goel and Sohi [19]

Remark 5.1.1. Lemma 3.3 provides that the subordinations in (5.1.1), (5.1.2) and (5.1.3) are N-subordinations, We will make use of this fact in obtaining our results.

5.2 In this section we first obtain some coefficient inequalities for $f(z)$ in $S_N^*(A,B)$. We will use the following lemma of Chapter II to prove our main theorem.

Lemma A. If m is a natural number such that $m \geq 2$, then

$$(5.2.1) \quad \frac{1}{(m!)^2} \{(A-B)^2 + \sum_{k=1}^{m-1} [(A-(Nk+1)B)^2-(Nk)^2] \prod_{j=0}^{k-1} u_j\} = \prod_{j=0}^{m-1} u_j$$

where $u_j = \frac{(A-(jN+1)B)^2}{N^2(j+1)^2}$, $j=0,1,2\ldots$ N is a fixed positive integer.

Theorem 5.2.1. Let $f(z) = z + \sum_{n=N+1}^{\infty} a_n z^n$ belong to $S_N^*(A,B)$. If $A-(N+1)B \leq N$, then

$$(5.2.2) \quad \sum_{n=t}^{N+t-1} (n-1)^2|a_n|^2 \leq (A-B)^2 \ , \ t \geq N+1.$$

If K is the smallest positive integer such that $A-B(NK+1) \leq NK$, then

$$(5.2.3) \quad \sum_{n=kN+1}^{(k+1)N} (n-1)^2|a_n|^2 \leq (kN)^2 \prod_{j=0}^{k-1} u_j \ , \text{ for } k \leq K.$$

If $k > K$, then

$$(5.2.4) \quad \sum_{n=kN+1}^{(k+1)N} (n-1)^2|a_n|^2 \leq (KN)^2 \prod_{j=0}^{K-1} u_j$$

Proof. Since $f(z)$ belongs to $S_N^*(A,B)$,

$$(5.2.5) \quad \frac{zf'(z)}{f(z)} = \frac{1+A\omega(z)}{1+B\omega(z)}$$

Remark 5.1.1 provides that $\omega(z)$ is a N-schwarz function i.e. $|\omega(z)| \leq |z^N|$. (5.2.5) provides

$$(z + \sum_{n=N+1}^{\infty} na_n z^n)(1+B\omega(z)) = (z + \sum_{n=N+1}^{\infty} a_n z^n)(1+A\omega(z)),$$

or

$$\sum_{n=N+1}^{\infty}(n-1)a_n z^n = \omega(z)\left((A-B)z + \sum_{n=N+1}^{\infty}(A-Bn)\, a_n z^n\right).$$

Since $\omega(z)$ has a zero of order N at $z=0$,

$$\sum_{n=N+1}^{N+t-1}(n-1)a_n z^n + \sum_{k=N+t}^{\infty}\alpha_k z^k = \omega(z)\left((A-B)z + \sum_{n=N+1}^{t-1}(A-Bn)a_n z^n\right), \quad t \geq N+1.$$

Since $|\omega(z)| \leq 1$ for $z \in U$, we have by Parseval's identity,

$$\sum_{n=N+1}^{N+t-1}(n-1)^2|a_n|^2 r^{2n} + \sum_{k=N+1}^{\infty}|\alpha_k|^2 r^{2k} \leq (A-B)^2 r^2 + \sum_{n=N+1}^{t-1}(A-Bn)^2|a_n|^2 r^{2n}$$

i.e.

$$\sum_{n=N+1}^{N+t-1}(n-1)^2|a_n|^2 r^{2n} \leq (A-B)^2 r^2 + \sum_{n=N+1}^{t-1}(A-Bn)^2|a_n|^2 r^{2n}, \quad t \geq N+1$$

Letting $r \to 1$ we get

$$\sum_{n=N+1}^{N+t-1}(n-1)^2|a_n|^2 \leq (A-B)^2 + \sum_{n=N+1}^{t-1}(A-Bn)^2|a_n|^2, \quad t \geq N+1.$$

Therefore,

$$(5.2.6) \qquad \sum_{n=t}^{N+t-1}(n-1)^2|a_n|^2 \leq (A-B)^2 + \sum_{n=N+1}^{t-1}\{(A-Bn)^2-(n-1)^2\}\ |a_n|^2.$$

If $A-B(N+1) < N$, then $(A-Bn) < n-1$ for $n \geq N+1$ and consequently

$$(5.2.7) \qquad |A-Bn| < n-1 \quad \forall\, n \geq N+1.$$

In view of (5.2.6) and (5.2.7), we have

$$\sum_{N=t}^{N+t-1} (n-1)^2|a_n|^2 \le (A-B)^2 \quad \text{for} \quad t \ge N+1$$

This proves (5.2.2).

Also, (5.2.6) provides

$$(5.2.8) \quad \sum_{n=kN+1}^{(k+1)N} (n-1)^2|a_n|^2 \le (A-B)^2 + \sum_{n=N+1}^{kN} \{(A-Bn)^2-(n-1)^2\}|a_n|^2 .$$

If $k \le K$, then

$$\sum_{n=kN+1}^{(k+1)N} (n-1)^2|a_n|^2 \le (A-B)^2 + \sum_{p=1}^{k-1} \{ \frac{(A-B(pN+1))^2-(pN)^2}{(pN)^2} \left(\sum_{n=pN+1}^{(p+1)N} (n-1)^2|a_n|^2 \right)\}$$

follows in view of the fact that $B < 0$ and $\frac{A-B(x+1)}{x}$ is a monotonically decreasing function of x.

Using lemma A, we get

$$\sum_{n=kN+1}^{(k+1)N} (n-1)^2|a_n|^2 \le (kN)^2 \prod_{j=0}^{k-1} u_j$$

where u_j as given in Lemma A.

If $k > K$, then

$$\sum_{n=kN+1}^{(k+1)N} (n-1)^2|a_n|^2 \le (KN)^2 \prod_{j=0}^{K-1} u_j$$

follows in view of the Lemma A and (5.2.8).

Bounds in (5.2.2) are sharp for the function

$$f(z) = \begin{cases} z(1+B\delta z^{(p+1)N})^{\frac{A-B}{(p+1)NB}}, & \text{for } B \neq 0 \\ z \exp \dfrac{A\delta z^{(p+1)N}}{(p+1)N}, & \text{for } B=0,\ |\delta| = 1. \end{cases}$$

Bounds in (5.2.3) are sharp for the function

$$f(z) = z(1+B\delta z^N)^{\frac{A-B}{BN}}, \quad |\delta| = 1.$$

<u>Corollary 5.2.1.</u> Let $f(z)$ belong to $S_N^*(\alpha)$. Then

$$\sum_{n=kN+1}^{(k+1)N} (n-1)^2|a_n|^2 \leq (kN)^2 \prod_{j=0}^{k-1} u_j$$

where $$u_j = \frac{(2(1-\alpha)+jN)^2}{N^2(j+1)^2}.$$

The estimates are sharp for the function $f_0(z) = \dfrac{z}{(1-z^N)^{\frac{2(1-\alpha)}{N}}}$.

<u>Remark.</u> We can easily get the coefficient estimates for $f(z) \in S_N^*(A,B)$ from (5.2.2), (5.2.3) and (5.2.4). Also, we see that the bounds in (5.2.2) and (5.2.3) are sharp for N-fold symmetric functions and in that case (5.2.2) and (5.2.3) give coefficient estimates directly.

5.3 In this section we study the class $Q_N(\alpha,A,B)$. We obtain a containment relation, integral representation and distortion properties for $f(z)$ belonging to $Q_N(\alpha,A,B)$. Also, we establish some coefficient estimates and a coefficient inequality for $f(z)$ in $Q_N(\alpha,A,B)$.

Theorem 5.3.1. $Q_N(\alpha,A,B) \subseteq Q_N(\beta,A,B)$, for $0 \leq \beta \leq \alpha$.

Proof. Let

$$J(\alpha,f(z)) = (1-\alpha)\,\frac{zf'(z)}{f(z)} + \alpha\left(1+\frac{zf''(z)}{f'(z)}\right)$$

One can easily show that if $f(z)$ is in $Q_N(\alpha,A,B)$, then $f(z)$ belongs to $S_N^*(A,B)$. Hence, if $f(z)$ is in $Q_N(\alpha,A,B)$, then

$$J(\alpha,f(z)) \underset{N}{<<} \frac{1+Az}{1+Bz}$$

and

$$J(0,f(z)) \underset{N}{<<} \frac{1+Az}{1+Bz} .$$

Every $J(\beta,f(z))$, $0 \leq \beta \leq \alpha$, can be written as a linear convex combination of $J(\alpha,f(z))$ and $J(0,f(z))$. Also, the image of $\frac{1+Az^N}{1+Bz^N}$ under $|z| \leq r < 1$ is convex. Hence the theorem follows by using the general definition of subordination.

In order to obtain the characterization for a function $f(z)$ in $Q_N(\alpha,A,B)$ $(\alpha > 0)$, we first prove the following theorem.

Theorem 5.3.2. Let $f(z)$ belong to $Q_N(\alpha,A,B)$. Consider

$$F_\beta(z) = f(z)\left[\frac{zf'(z)}{f(z)}\right]^\beta \quad (0 < \beta \leq \alpha) \tag{5.3.1}$$

If we choose a branch of $\left[\frac{zf'(z)}{f(z)}\right]^\beta$, which is equal to 1 when $z=0$, then the function $F_\beta(z)$ belongs to $S_N^*(A,B)$.

Proof. Let $f(z)$ belong to $Q_N(\alpha,A,B)$. Differentiating (5.3.1) logarithmically, we get

$$(5.3.2) \qquad \frac{zF'_\beta(z)}{F_\beta(z)} = (1-\beta)\frac{zf'(z)}{f(z)} + \beta\left(1+\frac{zf''(z)}{f'(z)}\right)$$

Since $f(z)$ is in $Q_N(\alpha,A,B)$, Theorem 5.3.2 provides that $f(z)$ belongs to $Q_N(\beta,A,B)$ for $0 < \beta \leq \alpha$. In view of (5.3.2), we have

$$\frac{zF'_\beta(z)}{F_\beta(z)} \underset{N}{<<} \frac{1+Az}{1+Bz}$$

Hence, $F_\beta(z)$ belongs to $S^*_N(A,B)$.

Now we consider the converse problem i.e. given the function $F(z)$ in $S^*_N(A,B)$, is the solution $f(z)$ of the differential equation

$$(5.3.3) \qquad F(z) = \left[\frac{zF'(z)}{F(z)}\right]^\alpha f(z)$$

With the boundary condition $f(0)=0$, a function of $Q_N(\alpha,A,B)$? We shall show that this indeed is the case.

Theorem 5.3.3. If $F(z)$ is in $S^*_N(A,B)$ and $\alpha > 0$, then the solution

$$(5.3.4) \qquad f(z) = \left[\frac{1}{\alpha}\int_0^z \frac{(F(\xi))^{1/\alpha}}{\xi}\, d\xi\right]^\alpha$$

of the differential equation (5.3.3) with the boundary condition $f(0)=0$ is a member of $Q_N(\alpha,A,B)$ where the branches are selected properly.

Proof. The proof of the theorem consists in showing that $f(z)$ is well defined analytic function in the unit disc U and belongs to $Q_N(\alpha, A, B)$. We follow the technique similar to that employed by Miller, Mocanu and Reade [58] . Let

$$(5.3.5) \qquad g(z) = z^{-1/\alpha} \int_0^z \frac{(F(\xi))^{1/\alpha}}{\xi} d\xi$$

where $F(z) = z + A_{N+1} z^{N+1} + \ldots$ belongs to $S_N^*(A,B)$. Also, (5.3.4) can be written as

$$(5.3.6) \qquad f(z) = z \left[\frac{1}{\alpha} g(z)\right]^{\alpha}$$

If we can show that $g(z)$ is independent of the path of integration, it will follow that $f(z)$ is well defined.

Since $F(z) = z[1 + A_{N+1} z^N + \ldots]$ belongs to $S_N^*(A,B)$ and thus being a member of a subclass of S^*, we have that $[1 + A_{N+1} z^N + \ldots]$ is nonzero in U.

Thus we may write

$$(5.3.7) \qquad (1 + A_{N+1} z^N + \ldots)^{1/\alpha} = 1 + \sum_{n=N}^{\infty} b_n z^n$$

as the power series expansion about $z=0$. From (5.3.7) it follows that

$$(5.3.8) \qquad \int_0^z \frac{(F(\xi))^{1/\alpha}}{\xi} d\xi = \alpha\, z^{1/\alpha} \left(1 + \sum_{n=N}^{\infty} \frac{b_n}{\alpha N + 1} z^n\right) + c$$

To obtain a solution of (5.5.8) which is analytic and zero at the origin, we take $c = 0$. Thus $g(z)$ is independent

of the path of integration, so that $f(z)$ given by (5.3.4) is well defined. Also from (5.3.4) we have by logarithmic differentiation.

$$J(\alpha,f(z)) = \frac{zF'(z)}{F(z)} \underset{N}{<<} \frac{1+Az}{1+Bz}.$$

Hence, $f(z)$ belongs to $Q_N(\alpha,A,B)$.

This completes the proof of the theorem.

Now we will prove a distortion theorem for $f(z)$ in $Q_N(\alpha,A,B)$.

Theorem 5.3.4. Let $f(z)$ belong to $Q_N(\alpha,A,B)$ $(\alpha > 0)$. Then

$$(5.3.9) \quad r\left[G\left(\frac{1}{\alpha N}, \frac{B-A}{BN\alpha}; \frac{1}{\alpha N}+1; Br^N\right)\right]^{\alpha} \leq |f(z)|$$

$$\leq r\left[G\left(\frac{1}{\alpha N}, \frac{B-A}{B\alpha N}; \frac{1}{\alpha N}+1; -Br^N\right)\right]^{\alpha}, \text{ for } B \neq 0$$

and

$$(5.3.10) \quad \left(\frac{1}{\alpha}\int_0^r \rho^{1/\alpha-1} e^{-\frac{A\rho^N}{N\alpha}} d\rho\right)^{\alpha} \leq |f(z)|$$

$$\leq \left(\frac{1}{\alpha}\int_0^r \rho^{1/\alpha-1} e^{\frac{A\rho^N}{N\alpha}} d\rho\right)^{\alpha}, \text{ for } B=0$$

where $G(a,b;c;z)$ is hypergeometric function.

Proof. We may take $z=r$ since the general case can be reduced to this by considering $\frac{f(\eta z)}{\eta}$ with suitably chosen η such that $|\eta| = 1$. By the integral representation for the function $f(z)$ in $Q_N(\alpha,A,B)$ $(\alpha > 0)$ there exists a function $F(z)$ in $S_N^*(A,B)$ such that

$$f(z) = \left[\frac{1}{\alpha} \int_0^z \frac{(F(\xi))^{1/\alpha}}{\xi} d\xi\right]^{\alpha} .$$

If we take $z=r$ and integrate along the positive real axis $\xi=\rho$, we get

$$(5.3.11) \qquad f(r) = \left[\frac{1}{\alpha} \int_0^z \frac{(F(\rho))^{1/\alpha}}{\rho} d\rho\right]^{\alpha} .$$

Since $F(z)$ belongs to $S_N^*(A,B)$,

$$(5.3.12) \qquad \frac{\rho}{(1-B\rho^N)^{\frac{B-A}{BN}}} \leq |F(\rho)| \leq \frac{\rho}{(1+B\rho^N)^{\frac{B-A}{BN}}}, \text{ for } B \neq 0$$

and

$$(5.3.13) \qquad \rho e^{-\frac{A\rho^N}{N}} \leq |F(\rho)| \leq \rho e^{\frac{A\rho^N}{N}}, \text{ for } B=0.$$

Let $B=0$. Using the R.H.S. of (5.3.13) in (5.3.11), we get

$$|f(r)| \leq \left(\frac{1}{\alpha} \int_0^r \rho^{1/\alpha-1} e^{\frac{A\rho^N}{N\alpha}} d\rho\right)^{\alpha}$$

This proves the R.H.S. of (5.3.10).

Let $B \neq 0$. Using (5.3.12) in (5.3.13), we get

$$|f(r)|^{1/\alpha} \leq \frac{1}{\alpha} \int_0^r \rho^{1/\alpha-1}(1+B\rho^N)^{\frac{A-B}{BN\alpha}} d\rho .$$

Setting $\rho^N = r^N u$, we get

$$N\rho^{N-1}d\rho = r^N du.$$

Therefore,

$$|f(r)|^{1/\alpha} \leq \frac{r^{1/\alpha}}{\alpha N} \int_0^1 u^{1/\alpha N-1}(1+Br^N u)^{\frac{A-B}{BN\alpha}} \, du$$

$$= r^{1/\alpha} G\left(\frac{1}{\alpha N}, \frac{B-A}{BN\alpha}; \frac{1}{\alpha N}+1; -Br^N\right).$$

Hence,

$$|f(r)| \leq r \left\{G\left(\frac{1}{\alpha N}, \frac{B-A}{BN\alpha}; \frac{1}{\alpha N}+1; -Br^N\right)\right\}^{\alpha}.$$

This proves the R.H.S. of (5.3.9).

To prove the L.H.S. of (5.3.9) and (5.3.10) we consider the straight line Γ joining 0 to $f(z)$. Since $f(z)$ is starlike Γ is the image of the jordan arc γ in U connecting 0 and z. The image of γ under the mapping $[f(z)]^{1/\alpha}$ will consist in general of many line segments enamating from the origin each of length

$$R^{1/\alpha} = |f(z)|^{1/\alpha} = \int_\gamma \left| d(f(\xi))^{1/\alpha}/d\xi \right| |d\xi|$$

From the integral representation of $f(z)$, there exists an $F(z)$ in $S_N^*(A,B)$ such that

$$\frac{d(f(\xi))^{1/\alpha}}{d\xi} = \frac{1}{\alpha} \frac{(F(\xi))^{1/\alpha}}{\xi}. \tag{5.3.13}$$

If $B=0$ and $\rho = |\xi|$, we deduce from (5.3.13)

$$R^{1/\alpha} = \frac{1}{\alpha} \int_\gamma \left| \frac{(F(\xi))^{1/\alpha}}{\xi} \right| |d\xi| \geq \frac{1}{\alpha} \int_0^r \rho^{1/\alpha-1} e^{\frac{A\rho^N}{\alpha N}} \, d\rho$$

Above inequality provides the L.H.S. of (5.3.10).

If $B \neq 0$, we deduce from (5.3.12)

$$R^{1/\alpha} \cdot \frac{1}{\alpha} \int_{\gamma} \left|\frac{(F(\xi))^{1/\alpha}}{\xi}\right| |d\xi| \geq \frac{1}{\alpha N} \int_{\gamma} \rho^{1/\alpha-1}(1-B\rho^N)^{\frac{A-B}{BN\alpha}} \, d\rho$$

$$= r^{1/\alpha} \{G(\frac{1}{\alpha N}, \frac{B-A}{BN\alpha}; \frac{1}{\alpha N}+1; Br^N)\} .$$

Hence, if $B \neq 0$

$$|f(z)| \geq r\{G(\frac{1}{\alpha N}, \frac{B-A}{BN\alpha}; \frac{1}{\alpha N}+1; Br^N)\}^{\alpha}$$

This proves the L.H.S. of (5.3.9).

The results are sharp for the function

$$f_0(z) = \begin{cases} (\frac{1}{\alpha} \int_0^z \xi^{1/\alpha-1}(1+B\xi^N)^{\frac{A-B}{BN\alpha}} d\xi)^{\alpha}, & \text{if } B \neq 0 \\ (\frac{1}{\alpha} \int_0^z \xi^{1/\alpha-1} e^{\frac{A\xi^N}{N\alpha}} d\xi)^{\alpha}, & \text{if } B = 0. \end{cases}$$

Remark 5.3.1. It is well known that a hypergeometric series $G(\alpha,\beta;\gamma;x)$ converges for $|z| < 1$ if $\gamma > 0$ and for $|z| \leq 1$ if $\alpha+\beta < \gamma$. Hence, if $f(z)$ is in $Q_N(\alpha,A,B)$ and $|B| < 1$, then $f(z)$ is bounded. If $B = -1$, then $f(z)$ in $Q_N(\alpha,A,B)$ is bounded for $\alpha > \frac{1+A}{N}$.

Remark 5.3.2. Let $f(z)$ belong to $Q_N(\alpha,A,B)$, $B \neq 0$, then the image of the unit disc under the mapping $W=f(z)$ always contains the disc $|W| \leq d(\alpha)$ where

$$d(\alpha) = \frac{1}{(1-B)^{\frac{B-A}{BN}}}, \text{ if } \alpha = 0,$$

$$d(\alpha) = \{G(\frac{1}{\alpha N}, \frac{B-A}{BN\alpha}, \frac{1}{\alpha N}+1; B)\}^{\alpha}, \text{ if } \alpha > 0.$$

Now we will obtain the upper bound $M(r) = \max\limits_{0\leq\theta<2\pi} f(re^{i\theta})$ for $f(z)$ in $Q_N(\alpha,A,B)$. We will use the following result [98] for proving our theorem.

Let $G(\alpha,\beta;\gamma;x)$ denote the hypergeometric function. If $\gamma-\alpha-\beta < 0$ and $x \to 1-0$, then

$$(5.3.14)\ (i)\quad \frac{G(\alpha,\beta;\gamma;x)}{\frac{\Gamma(\gamma)\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha)\Gamma(\beta)}\cdot(1-x)^{\gamma-\alpha-\beta}} \to 1$$

and if $\gamma-\alpha-\beta = 0$, then

$$(5.3.15)\quad \frac{G(\alpha,\beta;\gamma;x)}{\left(\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\log\left(\frac{1}{1-x}\right)\right)} \to 1$$

Theorem 5.3.5. Let $f(z)$ belong to $Q_N(\alpha,A,B)$ and $M(r) = \max\limits_{0\leq\theta<2\pi} f(re^{i\theta})$. If $|B| < 1$, then

$$(5.3.16)\quad M(r) \leq \begin{cases} r\left[G\left(\frac{1}{\alpha N}, \frac{B-A}{BN\alpha}; \frac{1}{\alpha N}+1; -Br^N\right)\right]^{\alpha}, & \text{if } B \neq 0, \\ \left(\frac{1}{\alpha}\int_0^r \rho^{1/\alpha-1} e^{\frac{A}{N\alpha}\rho^N} d\rho\right)^{\alpha}, & \text{if } B=0. \end{cases}$$

If $\alpha > \frac{1+A}{N}$ and $B = -1$, then

$$(5.3.17)\quad M(r) \leq r\left[G\left(\frac{1}{\alpha N}, \frac{B-A}{BN\alpha}; \frac{1}{\alpha N}+1; r^N\right)\right]^{\alpha}$$

If $B = -1$ and $\alpha \leq \frac{1+A}{N}$, then

$$(5.3.18)\quad M(r) \leq \left(\frac{1}{1-r^N}\right)^{\frac{1+A-\alpha N}{N}}, \text{ for } 0 \leq \alpha < \frac{1+A}{N}$$

and

$$(5.3.19)\qquad M(r) = \log\left(\frac{1}{1-r^N}\right)^{\frac{1+A}{N}}, \text{ for } \alpha = \frac{1+A}{N}.$$

Proof. (5.3.16) and (5.3.17) follow easily by using Theorem 5.3.4 and Remark 5.3.1.

If $B = -1$ and $0 < \alpha < \frac{1+A}{N}$, then (5.3.14) gives

$$(5.3.20)\qquad \lim_{r\to 1^-} \frac{G\left(\frac{1}{\alpha N}, \frac{1+A}{\alpha N}; \frac{1}{\alpha N}+1; r^N\right)}{(1-r^N)^{1-\frac{1+A}{\alpha N}}} = \frac{1}{(1+A-\alpha N)}.$$

Also, if $\alpha = 0$ then

$$M(r) \le \frac{r}{(1-r^N)^{\frac{1+A}{N}}}.$$

Hence, (5.3.18) follows easily by using (5.3.9) and (5.3.20).

If $B = -1$ and $\alpha = \frac{1+A}{N}$, then (5.3.15) provides

$$(5.3.21)\qquad \lim_{r\to 1^-} \frac{G\left(\frac{1}{\alpha N}, \frac{1+A}{\alpha N}; \frac{1}{\alpha N}+1; r^N\right)}{\log(1-r^N)} = \text{constant}$$

(5.3.19) follows by using (5.3.9) and (5.3.20).

This completes the proof of the theorem.

Now we will establish a coefficient inequality and some coefficient estimates for $f(z)$ in $Q_N(\alpha,A,B)$.

Theorem 5.3.6. Let $f(z) = z + \sum_{k=N+1}^{\infty} a_k z^k$ belong to $Q_N(\alpha,A,B)$ and μ be any complex number. If $N=1$, then

$$(5.3.22) \qquad |a_3 - \mu a_2^2| \leq \frac{A-B}{2(2\alpha+1)} \max \{1, |\nu|\}$$

where $\nu = \dfrac{2\mu(A-B)(2\alpha+1)+2(A-B)+(\alpha+1)[B(A+1)+3(B-A)]}{(\alpha+1)^2}$

For $n \geq 2$

$$(5.3.23) \quad |a_{N+2} - \mu a_{N+1}^2| \leq \frac{A-B}{(N+1)(1+\alpha(N+1))}$$

$$\{\max \; 1, |\mu| \frac{(A-B)^2(N+1)(1+\alpha(N+1))}{N^2(1+\alpha N)^2}\}.$$

Proof. Since $f(z) = z + \sum_{k=N+1}^{\infty} a_k z^k$ belongs to $Q_N(\alpha, A, B)$, there exists a N-Schwarz function $\omega(z)$ such that

$$(5.3.24) \qquad (1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1+\frac{zf''(z)}{f'(z)}\right) = \frac{1+A\omega(z)}{1+B\omega(z)} .$$

By expanding in power series both the sides of (5.3.24) and equating the coefficients of z^{N+1} and z^{N+2}, we have for $N=1$

$$(5.3.25) \qquad c_1 = \frac{a_2(\alpha+1)}{A-B}$$

$$c_2 = \frac{2a_3(2\alpha+1)}{(A-B)} + \frac{a_1^2\{2(A-B)+(\alpha+1)[B(\alpha+1)+3(B-A)]\}}{(A-B)^2}$$

and for $N \geq 2$

$$(5.3.27) \qquad c_N = \frac{a_{N+1}(\alpha N+1)N}{(A-B)} ,$$

$$(5.3.28) \qquad c_{N+1} = \frac{a_{N+2}(\alpha(N+1)+1)(N+1)}{(A-B)}$$

(5.3.25) and (5.2.26) with the Lemma due to Koegh and Merkes [38] provide

$$|c_2-\nu c_1^2| = \left|\frac{2a_3(2\alpha+1)}{A-B} + \frac{[2(A-B)+(\alpha+1)\{B(\alpha+1)+3(B-A)-\nu(\alpha+1)\}]\,a_2^2}{(A-B)^2}\right|$$

$$\leq \max\{1,|\nu|\}$$

Hence

$$\left|a_3 - \frac{(\alpha+1)[\nu(\alpha+1)-\{B(\alpha+1)+3(B-A)\}]-2(A-B)}{2(2\alpha+1)(A-B)}\,a_2^2\right|$$

$$\leq \frac{A-B}{2(2\alpha+1)}\max\{1,|\nu|\}.$$

Putting

$$\mu = \frac{(\alpha+1)[\nu(\alpha+1)-\{B(\alpha+1)+3(B-A)\}]-2(A-B)}{2(2\alpha+1)(A-B)}$$

we get (5.3.22).

(5.3.27) and (5.3.28) with lemma of Koegh and Merkes [38] provide

$$|c_{N+1}-\nu c_N^2| = \left|\frac{(N+1)(1+\alpha(N+1))}{(A-B)}\,a_{N+2} - \nu\,\frac{N^2(1+\alpha N)^2}{(A-B)^2}\,a_{N+1}^2\right|$$

$$\leq \max\{1,|\nu|\}.$$

Therefore,

$$|a_{N+2}-a_{N+1}^2| \leq \frac{A-B}{(N+1)(1+\alpha(N+1))} \max\{1, |\mu| \frac{(A-B)^2(N+1)(1+\alpha(N+1))}{N^2(1+\alpha N)^2}\}$$

This proves (5.2.23).

Corollary 5.3.1. Let $f(z)$ belong to $Q_N(\alpha, A, B)$. If $N=1$, then

$$(5.3.29) \qquad |a_2| \leq \frac{A-B}{\alpha+1}$$

and

$$(5.3.30) \qquad |a_3| \leq \frac{(A-B)[B(\alpha+1)^2+(B-A)(3\alpha+1)]}{2(2\alpha+1)(\alpha+1)^2}.$$

For $n \geq 2$,

$$(5.3.31) \qquad |a_{N+1}| \leq \frac{A-B}{N(1+\alpha N)}$$

and

$$|a_{N+2}| \leq \frac{A-B}{(N+1)(1+\alpha(N+1))}$$

Proof. Since $\omega(z) = \sum_{k=N}^{\infty} c_k z^k$ in (5.3.24) is a schwarz function

$$(5.3.33) \qquad |c_k \leq 1$$

(5.3.33) with (5.3.25) and (5.2.27) provide (5.3.29) and (5.3.31) respectively. (5.3.30) and (5.3.32) are obtained by putting $\mu=0$ in (5.3.22) and (5.3.23) respectively.

Corollary 5.3.2. Let $f(z) = z+\sum_{k=3}^{\infty} a_k z^k$ belong to $Q_2(\alpha, A, B)$

$$a_3| \leq \frac{A-B}{2(1+2\alpha)}$$

and

$$|a_4| \le \frac{A-B}{3(1+3\alpha)} .$$

Proof. Follows directly from Corollary 5.3.2.

5.4 In this section we study the class $R_N^\alpha(A,B)$. We obtain a coefficient inequality and coefficient estimates for $f(z)$ in $R_N^\alpha(A,B)$. Arc length of the image of $|z|=r<1$ and the area included by the image of $|z|=r<1$ under $f(z) \in R_N^\alpha(A,B)$ have been obtained. If $f(z)$ and $g(z)$ are in $R_N^\alpha(A,B)$, we obtain a condition on A,B,λ,α such that $(1-\lambda)z+\lambda(f(z)*zg'(z))$ is also in $R_N^\alpha(A,B)$. We first establish a lemma needed in obtaining coefficient estimates.

Lemma 5.4.1. For $f(z)$ in $R_N^\alpha(A,B)$, if

$$(5.4.1) \qquad f'(z) = p(z) = 1+\sum_{k=N}^{\infty} p_k z^k ,$$

then

$$(5.4.2) \qquad \sum_{k=t}^{N+t-1} |p_k|^2 \le (A-B)^2 \cos^2\alpha \; , \; k \ge N.$$

The bounds are sharp.

Proof. Since $f(z)$ belongs to $R_N^\alpha(A,B)$

$$f'(z) = \frac{1+(A\cos\alpha-iB\sin\alpha)\,e^{i\alpha}\omega(z)}{1+B\omega(z)}$$

where $\omega(z)$ has a zero of order N at $z=0$ and $|\omega(z)| \le |z^N|$ for $z \in U$.

Hence

$$(5.4.3) \qquad 1+\sum_{k=N}^{\infty} p_k z^k = \frac{1+(A\cos\alpha - i\,B\sin\alpha)e^{i\alpha}\omega(z)}{1+B\omega(z)}$$

Setting

$$s_t(z) = \sum_{k=N}^{N+t-1} p_k z^k$$

and

$$S_{t-1}(z) = (A-B)\cos\alpha\, e^{i\alpha} - B\sum_{k=N}^{t-1} p_k z^k ,$$

(5.4.3) can be written as

$$s_t(z) + \sum_{k=N+t}^{\infty} p_k z^k = \omega(z)\,[S_{t-1}(z) - \sum_{k=t}^{\infty} Bp_k z^k]$$

$$= \omega(z)\, S_{t-1}(z) + \sum_{k=t+N}^{\infty} \alpha_k z^k$$

or

$$s_t(z) + \sum_{k=N+t}^{\infty} (p_k-\alpha_k) z^k = \omega(z) S_{t-1}(z).$$

Since $|\omega(z)| < 1$, we have by Parseval's identity,

$$\sum_{k=N}^{N+t-1} |p_k|^2 r^{2k} + \sum_{k=N+t}^{\infty} |p_k-\alpha_k|^2 r^{2k} \le (A-B)^2\cos^2\alpha + B^2\sum_{k=N}^{t-1} |p_k|^2\, r^{2k}$$

Hence,

$$\sum_{k=N}^{N+t-1} |p_k|^2 r^{2k} \le (A-B)^2\cos^2\alpha + B^2\sum_{k=N}^{t-1} |p_k|^2\, r^{2k}.$$

Letting $r \to 1$, we get

$$\sum_{k=N}^{N+t-1} |p_k|^2 \leq (A-B)^2\cos^2\alpha + B^2 \sum_{k=N}^{t-1} |p_k|^2$$

or

$$\sum_{k=t}^{N+t-1} |p_k|^2 \leq (A-B)^2 \cos^2\alpha.$$

The result is sharp for the function

$$\rho_n(z) = \frac{1+(A\cos\alpha - iB\sin\alpha)\delta z^n}{1+B\delta z^n} \quad |\delta|=1, \; t\leq n\leq N+t-1.$$

Theorem 5.4.1. If $f(z)$ belongs to $R_N^{\alpha}(A,B)$ and

(5.4.4) $$f(z) = z + \sum_{k=N+1}^{\infty} b_k z^k ,$$

then

(5.4.5) $$|b_k| \leq \frac{(A-B)\cos\alpha}{n} \quad n \geq N+1$$

Proof. The result follows easily by using (5.4.4) and Lemma 5.4.1. The bounds are sharp for the function

$$f_{\alpha,n}(z) = \int_0^z \frac{1+(A\cos\alpha - iB\sin\alpha)e^{i\alpha}\delta\, s^n}{1+B\delta s^n}\, ds, \quad |\delta| = 1$$

Remark. If $f(z) = z + \sum_{k=N+1}^{\infty} b_k z^k$ belongs to $R_N^{\alpha}(A,B)$ and for $j \geq N+1$, $b_j = \frac{(A-B)\cos\alpha}{j}$, then the Corollary 5.4.1 guarentees that $b_k = 0$ for $k = j-N+1,\ldots j-1, j+1,\ldots j+N-1$.

Theorem 5.4.2. Let $f(z)$ given by (5.2.4) belong to $R_N^{\alpha}(A,B)$ and u be any complex number. Then for $N=1$

$$(5.4.6)\quad |b_3-ub_2^2| \le \frac{(A-B)\cos\alpha}{3} \max\{1, |B+\frac{3(A-B)}{4}\cos e^{i\alpha}u|\}$$

and for any $N \ge 2$

$$(5.4.7)\quad |b_{N+2}-ub_{N+1}^2| \le \frac{(A-B)\cos\alpha}{N+2} \max\{1, \frac{N+2}{(N+1)^2}(A-B)\cos\alpha|u|\}.$$

Proof. Since $f(z)$ belongs to $R_N^\alpha(A,B)$,

$$(5.4.8)\qquad f'(z) = \frac{1+(A\cos\alpha - iB\sin\alpha)\, e^{i\alpha}\omega(z)}{1+B\omega(z)}$$

where $|\omega(z)| \le |z^N|$ for $z \in U$.

Substituting $f'(z) = 1+\sum_{k=N}^{\infty}(k+1)b_{k+1}z^k$, $\omega(z) = \sum_{k=N}^{\infty}\alpha_k z^k$ and equating the coefficients of z^N and z^{N+1}, we get for $N=1$

$$(5.4.9)\quad \alpha_1 = \frac{2b_2}{(A-B)\cos\alpha\; e^{i\alpha}},$$

$$\alpha_2 = \frac{3}{(A-B)\cos\alpha\; e^{i\alpha}}\left[b_3+\frac{4B}{3(A-B)}\frac{b_2^2}{\cos\alpha\; e^{i\alpha}}\right].$$

For $N \ge 2$

$$\alpha_N = \frac{(N+1)b_{N+1}}{(A-B)\cos\alpha\; e^{i\alpha}},\quad \alpha_{N+1} = \frac{(N+2)b_{N+2}}{(A-B)\cos\alpha\; e^{i\alpha}}$$

(5.4.9) and (5.4.10) with the Lemma of Koegh and Merkes [38] provide (5.4.6) and (5.4.7) respectively.

Distortion theorems

Theorem 5.4.3. Let $f(z)$ belong to $R_N^{\alpha}(A,B)$. Then for $|z| = r < 1$,

$$(5.4.11)\quad |f'(z)| \leq [(\frac{1-ABr^{2N}}{1-B^2r^{2N}})^2 \cos^2\alpha + \sin^2\alpha]^{1/2} + \frac{(A-B)\cos\alpha\, r^N}{1-B^2r^{2N}}$$

$$(5.4.12)\quad |f'(z)| \geq [(\frac{1-ABr^{2N}}{1-B^2r^{2N}})^2 \cos^2\alpha + \sin^2\alpha]^{1/2} - \frac{(A-B)\cos\alpha\, r^N}{1-B^2r^{2N}}$$

and

$$(5.4.13)\quad |f(z)| \leq \int_0^r [\{(\frac{1-AB\xi^{2N}}{1-B^2\xi^{2N}})^2 \cos^2\alpha + \sin^2\alpha\}^{1/2} + \frac{(A-B)\cos\alpha\,\xi^N}{1-B^2\xi^{2N}}]d\xi$$

$$(5.4.14)\quad |f(z)| \geq \int_0^r [\{(\frac{1-AB\xi^{2N}}{1-B^2\xi^{2N}})^2 \cos^2\alpha + \sin^2\alpha\}^{1/2} - \frac{(A-B)\cos\alpha\,\xi^N}{1-B^2\xi^{2N}}]d\xi$$

Proof. (5.4.11) and (5.4.12) follow easily from (5.1.3) and Remark 5.1.1.

Now

$$f(z) = \int_0^z f'(z)dz = \int_0^r f'(se^{i\theta})e^{i\theta}ds$$

where $z = re^{i\theta}$. Then from (5.4.11) we have

$$|f(z)| \leq \int_0^r |f'(\xi e^{i\theta})e^{i\theta}|d\xi$$

$$\leq \int_0^r [\{(\frac{1-AB\xi^{2N}}{1-B^2\xi^{2N}})^2 \cos^2\alpha + \sin^2\alpha\}^{1/2} + \frac{(A-B)\cos\alpha\,\xi^N}{1-B^2\xi^{2N}}]\, d\xi$$

which yields (5.4.13).

Let z_0, $|z_0|=r$ be so chosen that $|f(z_0)| \leq |f(z)|$ for all z such that $|z|=r$. To find the lower bound of $|f(z)|$ we note that

$$|f(z)| = \int_L f'(z)dz = \int_L |f'(z)|\ |dz|$$

provided we integrate from 0 to z_0 along the path L (lying in $|z| < 1$) whose image is the line segment $[0, f(z_0)]$. On L, we have

$$|dz| \geq d|z|$$

and

$$|f'(z)| \geq [(\frac{1-ABr^{2N}}{1-B^2r^{2N}})^2 \cos^2\alpha + \sin^2\alpha]^{1/2} - \frac{(A-B)\cos\alpha\, r^N}{1-B^2r^{2N}} .$$

Hence,

$$|f(z)| \geq |f(z_0)| \geq \int_0^r [\{(\frac{1-AB\xi^N}{1-B^2\xi^{2N}})^2 \cos^2\alpha + \sin^2\alpha\}^{1/2}$$

$$- \frac{(A-B)\cos\alpha\ \xi^N}{1-B^2\xi^{2N}}]\, d\xi$$

This completes the proof of the theorem.

Following theorems can easily be proved.

Theorem 5.4.5. Let $f(z)$ belong to $R_N^\alpha(A,B)$. Then

$$|\arg f'(z)-\arg \frac{(1-B(A\cos\alpha - iB\sin\alpha)\, e^{i\alpha} r^{2N}}{1-B^2r^{2N}}|$$

$$\leq \sin^{-1} [\frac{A-B}{\{(1-ABr^{2N})^2\cos^2\alpha + \sin^2\alpha(1-B^2r^{2N})\}^{1/2}}] .$$

<u>Theorem 5.4.6.</u> If $f(z)$ and $g(z)$ belong to $R_N^{\alpha}(A,B)$, then $\lambda f(z) + (1-\lambda)g(z)$ also belongs to $R_N^{\alpha}(A,B)$, for $0 \leq \lambda \leq 1$.

<u>Arc length and Area of the image curve</u>

Now we obtain the length of the image of $|z|=r$ and the area bounded by the image of $|z|=r$ under $f(z)$ in $R_N^{\alpha}(A,B)$, $0 < r < 1$.

<u>Theorem 5.4.7.</u> Let $f(z)$ belong to $R_N^{\alpha}(A,B)$ and $L_r(f)$ denote the length of the image of $|z|=r$ under $f(z)$, $0<r<1$.

Then

$$L_r(f) \leq \begin{cases} 2\pi r \cos\alpha\left[1-\frac{A-B}{\pi B}\log\frac{1-Br^N}{1+Br^N}\right] + 2\pi r|\sin\alpha|, & \text{if } B \neq 0 \\ r\int_0^{2\pi}|1+A\cos\alpha\; r^N\; e^{i(N\theta+\alpha)}|d\theta, & \text{if } B=0, \end{cases}$$

<u>Proof.</u> Let

$$g(z)=f'(z) \text{ and } G(z) = \frac{1+(A\cos\alpha - iB\sin\alpha)e^{i\alpha}z}{1+Bz}.$$

Then (5.1.3) with Remark 5.1.1 provide

$$g(z) \underset{N}{\ll} G(z).$$

Using Lemma 1.3.3, we get

$$\int_0^{2\pi}|f'(re^{i\theta})|d\theta \leq \int_0^{2\pi}\left|\frac{1+(A\cos\alpha - iB\sin\alpha)e^{i(\alpha+N\theta)}r^N}{1+Br^N e^{iN\theta}}\right|d\theta.$$

Now

$$L_r(f) = \int_{|z|=r} |f'(z)||dz| = \int_0^{2\pi} |f'(re^{i\theta})| r\, d\theta$$

$$\leq \int_0^{2\pi} r \left| \frac{1+(A\cos\alpha - iB\sin\alpha) r^N e^{i(N\theta+\alpha)}}{1+Br^N e^{iN\theta}} - i\sin\alpha \right| d\theta$$

Since

$$\frac{A+B}{2B} - \frac{(1-B^2r^{2N})(A-B)}{2B(1+2Br^N\cos N\theta + B^2r^{2N})} = \operatorname{Re} \frac{1+Ar^N e^{iN\theta}}{1+Br^N e^{iN\theta}} \geq 0.$$

Also, $\cos\alpha > 0$ for $|\alpha| < \frac{\pi}{2}$ hence,

$$L_r(f) \leq r\left[\int_0^{2\pi} \frac{A+B}{2B}\cos\alpha\, d\theta - \int_0^{2\pi} \frac{A-B}{2B} \frac{(1-B^2r^{2N})}{(1+2Br^N\cos N\theta + B^2r^{2N})}\right] d\theta$$

$$+ \int_0^{2\pi} \left| \frac{A-B \cos\alpha\, r^N \sin N\theta}{1+2Br^N\cos N\theta + B^2r^{2N}} \right| d\theta + \int_0^{2\pi} |\sin\alpha|\, d\theta$$

$$\leq 2\pi r \cos\alpha \left[1 - \frac{A-B}{2\pi} \log \frac{1-Br^N}{1+Br^N}\right] + 2\pi r|\sin\alpha|, \text{ for } B \neq 0.$$

If $B=0$, then it is obvious that

$$L_r(f) \leq r \int_0^{2\pi} |1+A\cos\alpha\, r^N e^{i(N\theta+\alpha)}|\, d\theta.$$

This completes the proof of the theorem.

Theorem 5.4.8. Let $f(z)$ belong to $R_N^\alpha(A,B)$. If $A_r(f)$ denote the area included by the image of $|z|=r$ under $f(z)$, $0 < r < 1$, then

$$A_r(f) \le \begin{cases} \pi r^2\left[1 - \dfrac{(A-B)^2\cos^2\alpha}{B^2} + \dfrac{2(A-B)^2}{B^2 r^2}\displaystyle\int_0^r \dfrac{\rho d\rho}{1-B^2 r^{2N}}\right], & \text{for } B\neq 0 \\ \pi r^2\left[1 + \dfrac{A^2 r^{N+1}\cos^2\alpha}{N+1}\right] & \end{cases}$$

Proof. Let $f(z)$ belong to $R_N^{\alpha}(A,B)$. Then

$$f'(z) \underset{N}{\ll} \frac{1+(A\cos\alpha - iB\sin\alpha)e^{i\alpha}z}{1+Bz}.$$

Since

$$A_r(f) = \int_0^r\int_0^{2\pi} |f'(\rho e^{i\theta})|^2 \rho\, d\theta\, d\rho ,$$

Lemma 1.3.3 provides

$$A_r(f) \le \int_0^r\int_0^{2\pi}\left|\frac{1+(A\cos\alpha - iB\sin\alpha)e^{i\alpha}\rho^N e^{iN\theta}}{1+B\rho^N e^{iN\theta}}\right|^2 \rho\, d\theta\, d\rho$$

$$= \int_0^r\int_0^{2\pi} \frac{1+2(A\cos^2\alpha + B\sin^2\alpha)\rho^N\cos N\theta + (A^2\cos^2\alpha + B^2\sin^2\alpha)\rho^{2N} + 2(B-A)\cos\alpha\sin\alpha\,\rho^N\sin N\theta}{1+2B\rho^N\cos N\theta + B^2\rho^{2N}}\rho\, d\rho\, d\theta ,$$

If $B\neq 0$, then

$$A_r(f) \le 2\pi\int_0^r \left\{\frac{A\cos^2\alpha + B\sin^2\alpha}{B}\rho - \frac{A-B}{B}\,\frac{1-AB\rho^{2N}}{1-B^2\rho^{2N}}\rho\cos^2\alpha\right\} d\rho$$

$$= \pi r^2\left[1 - \frac{(A-B)^2\cos^2\alpha}{B^2} + \frac{2(A-B)^2}{B^2}\cos^2\alpha\int_0^{2\pi}\frac{\rho}{1-B^2\rho^{2N}}d\rho\right]$$

If $B=0$, then

$$A_r(f) \leq \int_0^r \int_0^{2\pi} |1+A\ \cos\alpha\ e^{i\alpha}\rho^N e^{iN\theta}|^2\ \rho\, d\theta\, d\rho$$

$$= \pi r^2 \left[1+ \frac{A^2 \cos^2\alpha\ r^{2N}}{N+1}\right].$$

This completes the proof of the theorem.

Convolution

Theorem 5.4.9. Let

$$f(z) = z + \sum_{B=N+1}^{\infty} b_k z^k$$

and

$$g(z) = z + \sum_{k=N+1}^{\infty} c_k z^k$$

belong to $R_N^{\alpha}(A,B)$. If

(5.4.16) $\qquad |B| + |\lambda|\ (A-B)\ \cos\alpha < 1,$

then

$$F(z) = z+\lambda \sum_{k=N+1}^{\infty} k\ b_k c_k z^k = (1-\lambda)z + \lambda(f(z)*zg'(z))$$

also belongs to $R_N^{\alpha}(A,B)$.

Proof. Since $f(z) \in R_N^{\alpha}(A,B)$,

$$f'(z) = \frac{1+(A\ \cos\alpha - iB\ \sin\alpha)e^{i\alpha}\omega(z)}{1+B\omega(z)}$$

where $\omega(z)$ is N-schwarz function and $|\omega(z)| \leq |z^N|$.

Hence,

(5.4.17) $$|f'(z)-b| \leq c$$

where $b = \dfrac{1-B(A\cos\alpha - iB\sin\alpha)\, e^{i\alpha} r^{2N}}{1-B^2 r^{2N}}$, $c = \dfrac{(A-B)\cos\alpha\, r^N}{1-B^2 r^{2N}}$.

One can easily check that $|1-b| \leq c$.

If $H(z) = \sum_{n=0}^{\infty} h_n z^n$ is a regular function in $|z|<1$ and $|H(z)| \leq M$, then by

(5.4.18) $$\sum_{n=0}^{\infty} |h_n|^2 \leq M^2.$$

Applying (5.4.18) in (5.4.17), we get

$$|1-b|^2 + \sum_{k=1}^{\infty} k^2 |b_k|^2 \leq c^2.$$

Hence,

(5.4.19) $$\sum_{k=N+1}^{\infty} b^2 |b_k|^2 \leq \frac{(A-B)^2 \cos^2\alpha\, r^{2N}}{1-B^2 r^{2N}}.$$

Similarly

(5.4.20) $$\sum_{k=N+1}^{\infty} k^2 |c_k|^2 \leq \frac{(A-B)^2 \cos^2\alpha\, r^{2N}}{1-B^2 r^{2N}}$$

Now

$$|F'(z)-b|^2 \leq |1-b+ \lambda \sum_{k=N}^{\infty} k^2 b_k c_k z^{k-1}|^2$$

$$\leq (1-b)^2 + 2|\lambda|\,|1-b| \sum_{k=N+1}^{\infty} k^2 |b_k|\,|c_k| r^{k-1}$$

$$+ |\lambda|^2 |\sum_{k=N+1}^{\infty} k^2\, b_k c_k z^{k-1}|^2$$

$$\leq (1-b)^2 + 2|\lambda||1-b|\Big(\sum_{k=N+1}^{\infty} k^2|b_k|^2 r^{k-1}\Big)^{1/2}\Big(\sum_{k=N+1}^{\infty} k^2|c_k|^2 r^{k-1}\Big)^{1/2}$$

$$+ |\lambda|^2\Big(\sum_{k=N+1}^{\infty} k^2|b_k|^2 r^{k-1}\Big)\Big(\sum_{k=N+1}^{\infty} k^2|c_k|^2 r^{k-1}\Big)$$

(by Cauchy-schwarz inequality)

$$\leq |1-b|^2 + 2|\lambda|\,|1-b|\frac{(A-B)^2\cos^2\alpha\ r^{2N}}{1-B^2r^{2N}} + |\lambda|^2\frac{(A-B)^2\cos^4\alpha\ r^{4N}}{(1-B^2r^{2N})^2}$$

$$= \frac{|B|^2\cos^2\alpha(A-B)^2r^{4N}}{(1-B^2r^{2N})^2} + \frac{2|\lambda||B|(A-B)^3\cos^3\alpha\ r^{4N}}{(1-B^2r^{2N})^2}$$

$$+ \frac{|\lambda|^2(A-B)^2\cos^4\alpha\ r^{4N}}{(1-B^2r^{2N})^2}$$

$$= \frac{(A-B)^2\cos^2\alpha\ r^{4N}}{(1-B^2r^{2N})^2}\Big[|B|+|\lambda|(A-B)\cos\alpha\Big]^2 \leq \frac{(A-B)^2\cos^2\alpha\ r^{4N}}{(1-B^2r^{2N})^2}$$

follows in view of (5.4.10).

Hence $F(z)$ belongs to $R_N^{\alpha}(A,B)$.

CHAPTER VI

EFFECT OF CERTAIN INTEGRAL OPERATORS ON A CLASS OF MEROMORPHIC FUNCTIONS

6.1 Let Σ denote the class of functions

$$(6.1.1) \qquad f(z) = z^{-1} + a_0 + a_1 z + a_2 z^2 + \ldots$$

regular in the punctured disc $0 < |z| < 1$ and Σ_0 denote the class of functions $f(z)$ in Σ with $a_0=0$.

Definition 6.1.1. A function $f(z)$ in Σ is said to belong to $\Sigma_K(A,B)$ $(-1 \leq B < A \leq 1)$ in $0 < |z| < 1$ if and only if

$$(6.1.2) \qquad -\{1+\frac{zf''(z)}{f'(z)}\} \ll \frac{1+Az}{1+Bz}, \quad \text{for } z \in U.$$

Definition 6.1.2. A function $f(z)$ in Σ is said to belong to $\Sigma^*(A,B)$ $(-1 \leq B < A \leq 1)$ in $0 < |z| < 1$ if and only if

$$(6.1.3) \qquad -\frac{zf'(z)}{f(z)} \ll \frac{1+Az}{1+Bz}, \quad \text{for } z \in U.$$

$\Sigma_K(1-2\lambda,-1) = \Sigma_K(\lambda)$ and $\Sigma^*(1-2\lambda,-1) = \Sigma^*(\lambda)$ $(0 \leq \lambda < 1)$ are well known subclasses of Σ consisting of functions meromorphically convex and starlike of order λ. The classes of functions meromorphically convex and starlike are identified by $\Sigma_K = \Sigma_K(0)$ and $\Sigma^* = \Sigma^*(0)$ respectively.

Definition 6.1.3. A function $f(z)$ in Σ is said to belong to $\Sigma_c(\lambda,\beta)$ $(0 \leq \lambda < 1)$, $(0 \leq \beta < 1)$, the class of close-to-conv

of order λ and type β in $0 < |z| < 1$, if there exists a function $g(z)$ in $\Sigma^*(\beta)$ such that

$$(6.1.4) \qquad -\mathrm{Re}\, \frac{zf'(z)}{g(z)} > \lambda \quad , \quad \text{for} \quad z \in U.$$

$\Sigma_c(\lambda,0) = \Sigma_c(\lambda)$ is the class of meromorphic close to-convex functions of order λ. It is well known that $\Sigma^*(A,B)$, $\Sigma_K(A,B)$ $(-1 \leq B < A \leq 1)$ consist of univalent functions while functions in $\Sigma_c(\lambda,\beta)$ need not be univalent.
Let $f(z) \in \Sigma$ and

$$(6.1.5) \qquad F(z) = c\int_0^1 u^c f(uz)du,\ 0 < u \leq 1,\ c > 0.$$

In section 6.2 we show that whenever $f(z)$ belongs to $\Sigma_{o,K}(0)$ or $\Sigma_o^*(0)$ i.e. $f(z)$ in Σ_K or Σ^* with $a_o=0$, then $F(z)$ belongs to $\Sigma_K(\frac{c}{c+2}, -\frac{c}{c+2})$ and $\Sigma^*(\frac{c}{c+2}, -\frac{c}{c+2})$ respectively. Further, we have shown that if $f(z)$ is close-to-convex of order λ with respect to $g(z)$ in Σ_o^*, then $F(z)$ is close-to-convex with respect to $G(z) = c\int_0^1 u^c f(uz)du$, $0 < u \leq 1$, $c > 0$. In section 6.3, the radius of convexity and starlikeness of $f(z)$ have been obtained whenever $F(z)$ is in $\Sigma_K(\frac{c}{c+2}, -\frac{c}{c+2})$ and $\Sigma^*(\frac{c}{c+2}, -\frac{c}{c+2})$ respectively. In Section 6.4 we obtain the radius of convexity and starlikeness of order λ of $f(z)$ whenever $F(z)$ is in $\Sigma_K(\lambda)$ and $\Sigma^*(\lambda)$ respectively. Radius of close-to-convexity of order λ and type β of $f(z)$ has been obtained, whenever $F(z)$ is close-to-convex of order λ and type β.

It has been noticed that the proof of Sohi and Goel [18] for proving $F(z)$ to be in Σ_c whenever $f(z)$ is in Σ_c $(c > 0)$ is incorrect. Moreover, their claim for $f(z)$ being meromorphically starlike in $0 < |z| < \sqrt{\frac{c}{c+2}}$ whenever $F(z)$ is in Σ^*, is seen wrong for some particular value of c.

6.2. In this section we first show that if $f(z)$ is $\Sigma_{0,K}(0)$ or $\Sigma_0^*(0)$, then $F(z)$ belongs to $\Sigma_{0,K}(\frac{c}{c+2}, -\frac{c}{c+2})$ and $\Sigma_0^*(\frac{c}{c+2}, -\frac{c}{c+2})$ respectively. Also, we show that if $f(z)$ is close-to-convex of order λ with respect to $g(z)$ in $\Sigma_0^*(0)$, then $F(z)$ is close-to-convex of order λ with respect to $G(z)$.

Theorem 6.2.1. Let $f(z) = z^{-1} + a_1 z + a_2 z^2 + \ldots$ belong to $\Sigma_0^*(0)$ and

$$(6.2.1) \qquad F(z) = c\int_0^1 u^c f(uz)du, \quad (0 < u \leq 1), \; c > 0.$$

If $F(z) \neq 0$ in $0 < |z| < 1$, then $F(z)$ is in $\Sigma_0^*(\frac{c}{c+2}, -\frac{c}{c+2})$. The result is sharp for the function $f_0(z) = \frac{1+z^2}{z}$.

Proof. Let $\omega(z)$ be a regular function defined in U

$$(6.2.2) \qquad -\frac{zF'(z)}{F(z)} = \frac{1+\frac{c}{c+2}\omega(z)}{1-\frac{c}{c+2}\omega(z)}.$$

Clearly, $\omega(z)$ has a zero of order 2 at $z=0$.

From the definition of $F(z)$, we have

$$(6.2.3) \qquad F(z)\left(\frac{zF'(z)}{F(z)}+c+1\right) = c\, f(z).$$

Using (6.2.2), we obtain from (6.2.3)

$$(6.2.4) \qquad f(z) = F(z)\left(\frac{1-\omega(z)}{1-\frac{c}{c+2}\,\omega(z)}\right)$$

Taking logarithmic derivative of (6.2.4) and then multiplying by $-z$, we get

$$(6.2.5) \qquad -\frac{zf'(z)}{f(z)} = \frac{1+\frac{c}{c+2}\,\omega(z)}{1-\frac{c}{c+2}\,\omega(z)} + \frac{2}{c+2}\,\frac{z\omega'(z)}{(1-\omega(z))(1-\frac{c}{c+2}\,\omega(z))}$$

Suppose there exists a $z_0 \in U$ such that $|\omega(z_0)|=1$ and $|\omega(z)| \leq 1$ for $|z| \leq |z_0|$. Then by [55]

$$(6.2.6) \qquad z_0\omega'(z_0) = m\,\omega(z_0), \quad \text{for some } m \geq 2.$$

Using (6.2.6) in (6.2.5), we get

$$-\frac{z_0f'(z_0)}{f(z_0)} = \frac{1+\frac{c}{c+2}\,\omega(z_0)}{1-\frac{c}{c+2}\,\omega(z_0)} + \frac{2m}{c+2}\,\frac{\omega(z_0)}{(1-\omega(z_0))(1-\frac{c}{c+2}\,\omega(z_0))}$$

Putting $\omega(z_0) = e^{i\theta}(0 \leq \theta < 2\pi)$, the above expression reduces to

$$-\frac{z_0f'(z_0)}{f(z_0)} = \frac{1+\frac{c}{c+2}\,e^{i\theta}}{1-\frac{c}{c+2}\,e^{i\theta}} - \frac{m/2}{(c+2)\sin^2\frac{\theta}{2}}\,\frac{1-e^{i\theta}}{1-\frac{c}{c+2}\,e^{i\theta}},$$

Taking real part of both the sides (6.2.7),

$$-\mathrm{Re}\,\frac{z_0 f'(z_0)}{f(z_0)} \leq \frac{1-(\frac{c}{c+2})^2}{1-\frac{2c}{c+2}\cos\theta + (\frac{c}{c+2})^2}$$

$$-\frac{1}{(c+2)\sin\frac{2\theta}{2}}\;\frac{1-\frac{2(c+1)}{c+2}\cos\theta + (\frac{c}{c+2})}{(1-\frac{2c}{c+2}\cos\theta + (\frac{c}{c+2})^2)}$$

which is a contradiction to the given hypothesis that $f(z)$ is in $\Sigma_0^*(0)$. Hence, $|\omega(z)|<1$ for $|z| < 1$ and therefore, $F(z)$ belongs to $\Sigma_0^*(\frac{c}{c+2}, -\frac{c}{c+2})$.

Following the lines of proof of Theorem 6.2.1, we can also prove the following theorem.

Theorem 6.2.2. Let $f(z) = z^{-1} + a_1 z + a_2 z^2 + \ldots$ belongs to $\Sigma_{0,K}(0)$ and

$$F(z) = c\int_0^1 u^c f(uz)\,du\ , \quad (0 < u \leq 1),\ c > 0.$$

If $F(z) \neq 0$ for $0 < |z| < 1$, then $F(z)$ belongs to $\Sigma_{0,K}(\frac{c}{c+2}, -\frac{c}{c+2})$.

Now we will prove that $f(z)$ close-to-convex of order with respect to $g(z)$ in $\Sigma_0^*(0)$, then $F(z)$ is close-to-convex of order λ with respect to $G(z)$.

Theorem 6.2.3. Let

$$f(z) = z^{-1} + a_0 + a_1 z + \ldots$$

$$g(z) = z^{-1} + b_1 z + b_2 z^2 \ldots$$

be regular in $0 < |z| < 1$ and $f(z)$ be close-to-convex of order λ with respect to $g(z)$.

Let

$$F(z) = c\int_0^1 u^c f(uz)du,\ G(z) = c\int_0^1 u^c g(uz)du,\quad (0 < u \leq 1),\ c > 0.$$

If $G(z) \neq 0$ for $0 < |z| < 1$, then $F(z)$ is close-to-convex of order λ with respect to $G(z)$.

Proof. Since $g(z)$ belongs to Σ_0^*, by Theorem 6.2.1 $G(z)$ belongs to $\Sigma_0^*(\frac{c}{c+2}, -\frac{c}{c+2})$. Therefore, it is sufficient to show that

$$-\mathrm{Re}\ \frac{zF'(z)}{G(z)} > \lambda\ ,\ \text{for}\ \ z \in U.$$

Define a regular function $\omega(z)$ in U by

$$(6.2.8)\qquad -\frac{zF'(z)}{G(z)} = \frac{1+(1-2\lambda)\omega(z)}{1-\omega(z)}$$

Clearly $\omega(0) = 0$.

From the definition of $F(z)$ and (6.2.8), we have

$$(6.2.9)\qquad cf(z) = (1+c)F(z) - \frac{1+(1-2\lambda)\omega(z)}{1-\omega(z)}\,G(z).$$

Differentiating both the sides of (6.2.9) and then multiplying by $-z$, we get

$$(6.2.10)\quad -czf'(z) = -(1+c)zF'(z) - \frac{2(1-\lambda)zG'(z)}{(1-\omega(z))^2} + \frac{1+(1-2\lambda)\omega(z)}{1-\omega(z)}\,zG'(z).$$

Also, from the definition of $G(z)$

$$(6.2.11) \qquad cg(z) = (1+c)G(z) + zG'(z).$$

Hence, (6.2.10) and (6.2.11) provides

$$(6.2.12) \quad -\frac{zf'(z)}{g(z)} = \frac{1+(1-2\lambda)\omega(z)}{1-\omega(z)} + \frac{2(1-\lambda)z\omega'(z)}{(1-\omega(z))^2} \cdot \frac{1}{(1+c+\frac{zG'(z)}{G(z)})}.$$

We claim that $|\omega(z)| < 1$, for $z \in U$.

Suppose there exists a $z_0 \in U$ such that $|\omega(z_0)| = 1$ and $|\omega(z)| \leq 1$ for $|z| \leq |z_0|$. By the lemma of Jack [30]

$$z_0\omega'(z_0) = m\omega(z_0) \;, \quad \text{for some} \quad m \geq 1.$$

The use of Jack's lemma in (6.2.12) provides

$$-\frac{zf'(z_0)}{g(z_0)} = \frac{1+(1-2\lambda)\omega(z_0)}{1-\omega(z_0)} + \frac{2(1-\lambda)\omega(z_0)}{(1-\omega(z_0))^2} \cdot \frac{1}{(1+c+\frac{z_0G'(z_0)}{G(z_0)})}$$

Since $G(z)$ belongs to $\Sigma_0^*(\frac{c}{c+2}, -\frac{c}{c+2})$, $\text{Re}\{1+c+\frac{z_0G'(z_0)}{G(z_0)}\} > 0$, for $z \in U$.

Putting $\omega(z_0) = e^{i\theta}$, $(0 \leq \theta < 2\pi)$, we get

$$\text{Re}\, \frac{z_0f'(z_0)}{g(z_0)} < \lambda$$

which is a contradiction to the given hypothesis that $f(z)$ is close-to-convex of order λ with respect to $g(z)$. Hence, $|\omega(z)| < 1$ and therefore $f(z)$ belongs to $\Sigma_c(\lambda)$.

This completes the proof of the theorem.

Remark 6.2.1. Sohi and Goel [18] proved that if $f(z)$ in Σ_c, then $F(z)$ belongs to Σ_c for $0 < |z| < 1$. But we are doubtful about their claim because the inequality {(16), Page 23}

$$\left|h(z) - \frac{1+c-a}{(1+c-a)^2-d^2}\right| \leq \frac{d}{(1+c-a)^2-d^2}$$

is true only if $(1+c-a)-d$ is positive. They used it without observing this restriction.

6.3 In this section we obtain the radius of convexity and starlikeness of $f(z)$, whenever $F(z)$ is in $\Sigma^*(\frac{c}{c+2}, -\frac{c}{c+2})$ and $\Sigma_K(\frac{c}{c+2}, -\frac{c}{c+2})$ respectively.

Theorem 6.3.1. Let $F(z) = z^{-1} + b_0 + b_1 z + \ldots$ belong to $\Sigma^*(\frac{c}{c+2}, -\frac{c}{c+2})$. If

$$f(z) = \frac{1}{c}\left[(c+1)F(z) + zF'(z)\right], \quad c > 0,$$

then $F(z)$ belongs to Σ^* for $0 < |z| < \sqrt{\frac{c+2}{c+4}}$.

We shall need the following lemma due to Singh and Goel [70], for proving our result.

Lemma 6.3.1. Let $\omega(z)$ be regular in U and satisfies the conditions (i) $\omega(0) = 0$ (ii) $|\omega(z)| < 1$ for $z \in U$, then

$$(6.3.1) \qquad |z\omega'(z) - \omega(z)| \leq \frac{|z|^2 - |\omega(z)|^2}{1-|z|^2}.$$

Proof of the theorem. Since $F(z)$ belongs to $\Sigma^*(\frac{c}{c+2}, -\frac{c}{c+2})$, there exists a function $\omega(z)$ regular in U with $\omega(0)=0$,

$|\omega(z)| < 1$ such that

$$(6.3.2) \qquad -\frac{zF'(z)}{F(z)} = \frac{1+\frac{c}{c+2}\,\omega(z)}{1-\frac{c}{c+2}\,\omega(z)} .$$

This yields

$$f(z) = \frac{1-\omega(z)}{1-\frac{c}{c+2}\omega(z)}\,F(z) = \frac{1-\omega(z)}{1-b\omega(z)}\,F(z) \quad \text{where } b = \frac{c}{c+2} .$$

Taking the logarithmic derivative of both the sides of (6.3.2), we get

$$(6.3.3) \quad -\frac{zf'(z)}{f(z)} = \frac{1+\frac{c}{c+2}\,\omega(z)}{1-\frac{c}{c+2}\,\omega(z)} + (1-b)\,\frac{z\omega'(z)}{(1-b\omega(z))(1-\omega(z))} .$$

Using (6.3.1), we get from (6.3.3)

$$-\mathrm{Re}\,\frac{zf'(z)}{f(z)} \geq \mathrm{Re}\,\frac{1+\frac{c}{c+2}\omega(z)}{1-\frac{c}{c+2}\omega(z)} + (1-b)\left[\mathrm{Re}\,\frac{\omega(z)}{(1-\omega(z))(1-b\omega(z))} - \frac{r^2-|\omega(z)|^2}{(1-r^2)(1-\omega(z))(1-b\omega(z))}\right]$$

$$= \frac{1}{1-b}\,\mathrm{Re}[p(z) - \mathrm{Re}\,\frac{b}{p(z)} + \frac{|1-p(z)|^2 - r^2|p(z)-b|^2}{(1-r^2)|p(z)|}]$$

where $p(z) = \frac{1-b\omega(z)}{1-\omega(z)}$.

It is easy to see that the transformation $p(z) = \frac{1-b\omega(z)}{1-\omega(z)}$ maps the circle $|\omega(z)| \leq r$ onto the circle

$$|p(z)-a| \leq d \quad \text{where} \quad a = \frac{1-br^2}{1-r^2},\ d = \frac{r(1-b)}{1-r^2} .$$

If we put $p(z) = Re^{i\theta}$ and denote the R.H.S. of (6.3.4) by $S(R,\theta)$, then

$|\omega(z)| < 1$ such that

$$(6.3.2) \qquad -\frac{zF'(z)}{F(z)} = \frac{1+\frac{c}{c+2}\,\omega(z)}{1-\frac{c}{c+2}\,\omega(z)} .$$

This yields

$$f(z) = \frac{1-\omega(z)}{1-\frac{c}{c+2}\omega(z)}\,F(z) = \frac{1-\omega(z)}{1-b\omega(z)}\,F(z) \quad \text{where } b = \frac{c}{c+2} .$$

Taking the logarithmic derivative of both the sides of (6.3.2), we get

$$(6.3.3) \quad -\frac{zf'(z)}{f(z)} = \frac{1+\frac{c}{c+2}\,\omega(z)}{1-\frac{c}{c+2}\,\omega(z)} + (1-b)\,\frac{z\omega'(z)}{(1-b\omega(z))(1-\omega(z))} .$$

Using (6.3.1), we get from (6.3.3)

$$-\text{Re}\,\frac{zf'(z)}{f(z)} \geq \text{Re}\,\frac{1+\frac{c}{c+2}\omega(z)}{1-\frac{c}{c+2}\omega(z)} + (1-b)\,[\text{Re}\,\frac{\omega(z)}{(1-\omega(z))(1-b\omega(z))} - \frac{r^2-|\omega(z)|^2}{(1-r^2)(1-\omega(z))(1-b\omega(z))}$$

$$= \frac{1}{1-b}\,\text{Re}[p(z)-\text{Re}\,\frac{b}{p(z)} + \frac{|1-p(z)|^2-r^2|p(z)-b|^2}{(1-r^2)|p(z)|}]$$

where $p(z) = \frac{1-b\omega(z)}{1-\omega(z)}$.

It is easy to see that the transformation $p(z) = \frac{1-b\omega(z)}{1-\omega(z)}$ maps the circle $|\omega(z)| \leq r$ onto the circle

$$|p(z)-a| \leq d \quad \text{where} \quad a = \frac{1-br^2}{1-r^2},\ d = \frac{r(1-b)}{1-r^2} .$$

If we put $p(z) = Re^{i\theta}$ and denote the R.H.S. of (6.3.4) by $S(R,\theta)$, then

(6.3.5) $$S(r,\theta) = \frac{1}{(1-b)}\left[\left(R-\frac{b}{R}-2a\right)\cos\theta + R+(a^2-d^2)R^{-1}\right]$$

where $a-d \le R \le a+d$ and $\cos\theta \ge \frac{R^2+a^2-d^2}{2Ra}$.

Let $T(R) = 2a-R+\frac{b}{R}$.

If $T(R) \le 0$, then obviously $S(R,\theta) > 0$ inside the disc $|p(z)-a| \le d$. If $T(R) > 0$, then minimum of $S(R,\theta) > 0$ w.r.t. θ inside the $|p(z)-a| \le d$ is attained at $\theta=0$.

Putting $\theta=0$ in (6.3.5), we obtain

(6.3.6) $$L(R) \equiv S(R,0) = \frac{1}{(1-b)}\left[2R+\frac{-b+a^2-d^2}{R}-2a\right]$$

Minimum of $L(R)$ w.r.t. R is attained at $R = \sqrt{\frac{-b+a^2-d^2}{2}}$.

If $a-d \ge \sqrt{\frac{-b+a^2-d^2}{2}}$, then the minimum of $L(R)$ w.r.t. R is attained at $R = a-d$ and

$$L(a-d) = \frac{1-br^2}{(1+br)(1+r)} > 0, \text{ for } r < 1.$$

If $a-d < \sqrt{\frac{-b+a^2-d^2}{2}}$, then the minimum of $L(R)$ w.r.t. R is attained at $R = \sqrt{\frac{-b+a^2-d^2}{2}}$.

Also,

$$L\left(\sqrt{\frac{-b+a^2-d^2}{2}}\right) = 2\sqrt{2(-b+a^2-d^2)} - 2a \ge 2(a-2d) \ge 0, \text{ for}$$

$$\text{for } r < \frac{-4+\sqrt{c^2+4c+4}}{c}.$$

Now we only have to show that L(R) is positive for $\frac{-4+\sqrt{c^2+2c+4}}{c} \le r < \sqrt{\frac{c+2}{c+4}}$. It is sufficient to show that

(6.3.8) $Q(r) = -\{r^4(2-b)+2\{(a-b)^2-b\}r^2-(1-2b)\} > 0$ for

$$\frac{-4+\sqrt{c^2+2c+4}}{c} \le r < \sqrt{\frac{c+2}{c+4}} .$$

The equation

$$r^4(2-b)b + 2\{(1-b)^2-b\}r^2-(1-2b) = 0$$

has four roots $\pm\sqrt{\frac{c-2}{c}}$, $\pm\sqrt{\frac{c+2}{c+4}}$.

If $c < 2$, then one pair consists of imaginary roots and for $c \ge 2$

$$\sqrt{\frac{c+2}{c+4}} \ge \frac{-4+\sqrt{c^2+4c+4}}{c} \ge \sqrt{\frac{c-2}{c}} .$$

Also,

$$\left.\frac{\partial\theta}{\partial r}\right|_{r=\sqrt{\frac{c+2}{c+4}}} < 0$$

gives that L(R) is positive for $\frac{-4+\sqrt{c^2+4c+4}}{c} \le r < \sqrt{\frac{c+2}{c+4}}$. (6.3.7) with the above argument gives that L(R) is positive for $r < \sqrt{\frac{c+2}{c+4}}$. This completes the proof of the theorem.

Theorem 6.3.2. Let $F(z) = z^{-1}+b_0+b_1z+\dots$ belong to $\Sigma_K(\frac{c}{c+2}, -\frac{c}{c+2})$ and

$$f(z) = \frac{1}{c}\,[(c+1)F(z) + zF'(z)] , \quad c > 0.$$

Then $f(z)$ also belongs to Σ_K for $0 < |z| < \sqrt{\frac{c+2}{c+4}}$.

Proof. The above theorem can be proved on the same lines as the proof of the Theorem 6.3.2.

6.4 It can be easily shown that whenever $f(z)$ belongs to $\Sigma^*(\lambda)$ or $\Sigma_K(\lambda)$, $F(z)$ also belongs to $\Sigma^*(\lambda)$ and $\Sigma_K(\lambda)$ respectively. In this section we obtain the radius of convexity and starlikeness of order λ of $f(z) = \frac{1}{c}\left[(c+1)F(z) + zF'(z)\right]$, $c > 0$, whenever $F(z)$ belongs to $\Sigma_K(\lambda)$ and $\Sigma^*(\lambda)$ respectively. We obtain the radius of close-to-convexity of order λ and type β of $f(z)$ whenever $F(z)$ is close-to-convex of order λ and type β. Also, we show that the claim of Sohi and Goel [18] that $f(z)$ belongs to Σ^* in $0 < |z| < \sqrt{\frac{c}{c+2}}$ whenever $F(z)$ belongs to Σ^* is incorrect.

Theorem 6.4.1. Let $F(z) = z^{-1}+b_0+b_1z+\ldots$ belong to $\Sigma^*(\lambda)$ and

$$f(z) = \frac{1}{c}\left[(c+1)F(z) + zF'(z)\right], \quad c > 0$$

Then $f(z)$ is meromorphically starlike of order λ

in $0 < |z| < \frac{c}{c+2-2\lambda}$, if $0 \leq \lambda \leq \frac{1}{2}$

in $0 < |z| < \frac{-\lambda+\sqrt{\lambda^2+c(c+2-2\lambda)}}{c+2-2\lambda}$, if $\frac{1}{2} \leq \lambda < 1$.

The result is sharp.

Proof: Since $F(z)$ belongs to $\Sigma^*(\lambda)$, there exists a function $\omega(z)$ regular in U with $\omega(0)=0$ and $|\omega(z)| < 1$, such that

$$(6.4.1) \qquad -\frac{zF'(z)}{F(z)} = \frac{1+(1-2\lambda)\omega(z)}{1-\omega(z)}$$

This yields

$$(6.4.2) \qquad f(z) = \frac{1-b\omega(z)}{1-\omega(z)} F(z) \quad \text{where} \quad b = \frac{c+2-2\lambda}{c} .$$

Taking the logarithmic derivative of both the sides of (6.4.2) and multiplying by $-z$, we get

$$(6.4.3) \qquad -\frac{zf'(z)}{f(z)} = (1-\lambda)\frac{1+\omega(z)}{1-\omega(z)} - (b-1)\frac{z\omega'(z)}{(1-b\omega(z))(1-\omega(z))} .$$

Using (6.3.1), we get from (6.4.2)

$$-\text{Re}\frac{zf'(z)}{f(z)} - \lambda \geq (1-\lambda)\frac{1+\omega(z)}{1-\omega(z)} - (b-1)\Big[\text{Re}\frac{\omega(z)}{(1-b\omega(z))(1-\omega(z))}$$

$$- \frac{r^2-|\omega(z)|^2}{(1-r^2)(1-b\omega(z))(1-\omega(z))}$$

$$(6.4.4) \quad \geq \frac{1}{(b-1)}\Big[(2\lambda-1)\text{Re }h(z) + \text{Re}\frac{b}{h(z)} - \frac{|1-h(z)|^2 - r^2|h(z)-b|^2}{(1-r^2)|h(z)|}\Big]$$

where $h(z) = \dfrac{1-b\omega(z)}{1-\omega(z)}$.

The transformation $h(z) = \dfrac{1-b\omega(z)}{1-\omega(z)}$ maps the circle $|\omega(z)| \leq r$ onto the circle

$$(6.4.5) \quad |h(z)-a| \leq d \quad \text{where} \quad a = \frac{1-br^2}{1-r^2}, \quad d = \frac{(b-1)r}{1-r^2} .$$

If we put $h(z) = Re^{i\theta}$, then $a-d \leq R \leq a+d$ and that for $r < \frac{1}{b}$, $\cos\theta > 0$. Let $S(R,\theta)$ denote the R.H.S. of (6.4.4) then

$$(6.4.6)\quad S(R,\theta) = \frac{1}{(b-1)}\left[((2\lambda-1)R + \frac{b}{R} - 2a)\cos\theta + (a^2-d^2)R^{-1} - (b+1)\lambda + R\right]$$

Suppose $0 \le \lambda \le \frac{1}{2}$ and

$$(2\lambda-1)R + \frac{b}{R} - 2a \ge 0.$$

Also,

$$(6.4.7)\quad \cos\theta \ge \frac{R^2+a^2-d^2}{2Ra}$$

with (6.4.4) provides

$$(6.4.8)\quad S(R,\theta) \ge \frac{1}{(b-1)}\left[((2\lambda-1)R + \frac{b}{R})\frac{R^2+a^2-d^2}{2Ra} - (b+1)\lambda\right]$$

we will show that for $r < \frac{1}{b}$, R.H.S. of (6.4.8) is positive. Since $2\lambda-1 \le 0$ and for $r < \frac{1}{b}$, $a^2-d^2 > 0$. Hence, the R.H.S. of (6.4.8) is a decreasing function of R and therefore, the minimum of R.H.S. of (6.4.8) with respect to R is attained at $R = a+d$.

The above argument provides that

$$S(R,\theta) \ge \frac{1}{(b-1)a}\left[\left(\frac{b}{a+d} + (2\lambda-1)(a+d)\right)\frac{(a+d)^2+a^2-d^2}{2(a+d)} - (b+1)a\lambda\right]$$

$$(6.4.9)\quad = \frac{(1-\lambda)(1-b\,r^2)}{(1+br)(1+r)} - \lambda(b-1)r.$$

R.H.S. of (6.4.9) is positive for $r < \frac{1}{b}$.

Suppose

$$(6.4.10)\quad \frac{b}{R} + (2\lambda-1)R - 2a < 0$$

(6.4.6) provides

$$\frac{\partial S}{\partial \theta} = \frac{1}{(b-1)}\, T(R) \sin\theta$$

where $T(R) = -(\frac{b}{R} + (2\lambda-1)R-2a) > 0$, (follows in view of (6.4.10))

Hence, the minimum of $S(R,\theta)$ w.r.t. θ is attained at $\theta = 0$. Let it be $L(R)$. Then

$$L(R) = S(R,0) = \frac{1}{(b-1)}\left[2\lambda R + \frac{b}{R} - 2a + (a^2-d^2)R^{-1} - (b+1)\lambda\right]$$

and therefore,

$$\frac{\partial L}{\partial R} = \frac{1}{(b-1)}\left[2\lambda - \frac{a^2-d^2+b}{R^2}\right]$$

Hence, $L(R)$ is a decreasing function of R for $0 < R \leq \sqrt{\frac{a^2-d^2+b}{2\lambda}}$.

If $a+d \geq \sqrt{\frac{a^2-d^2+b}{2\lambda}}$, then the minimum of $L(R)$ with respect to R is attained at $R = a+d$ and

$$\text{(6.4.11)} \qquad L(a+d) = \frac{(1-\lambda)(1-br^2) - \lambda(b-1)r}{(1+br)(1+r)}.$$

R.H.S. of (6.4.11) is the same as the R.H.S. of (6.4.9). Hence for $r < \frac{1}{b}$, it is positive. Also, it can be easily checked that for $r < \frac{1}{b}$ and $0 \leq \lambda \leq \frac{1}{2}$,

$$(a+d) \leq \sqrt{\frac{a^2-d^2+b}{2\lambda}}.$$

Thus we proved that if $F(z)$ belongs to $\Sigma^*(\lambda)$, $0 \leq \lambda \leq \frac{1}{2}$, then $f(z)$ is meromorphically starlike of order λ in $0 < |z| < \frac{c}{c+2-2\lambda}$.

Next, we consider the case when $\frac{1}{2} < \lambda < 1$.

Suppose

$$(2\lambda-1)R + \frac{b}{R} - 2a \geq 0$$

(6.4.6) with (6.4.7) provides

$$S(R,\theta) \geq \frac{1}{(b-1)}\left[\left((2\lambda-1)R + \frac{b}{R}\right)\frac{R^2+a^2-d^2}{2Ra} - (b+1)\lambda\right]$$

$$(6.4.12) \quad = \frac{1}{2a(b-1)}\Big[b+(2\lambda-1)(a^2-d^2) + \frac{b(a^2-d^2)}{R^2}$$

$$+ (2\lambda-1)R^2-2(b+1)a\Big] .$$

The R.H.S. of (6.4.12) is a decreasing function of R for $0 \leq R^2 \leq \sqrt{\frac{b(a^2-d^2)}{2\lambda-1}}$. If

$$(a+d) \leq \left[\frac{b(a^2-d^2)}{2\ -1}\right]^{1/2}$$

then the minimum of $S(R,\theta)$ with respect to R is attained at $R = a+d$ and therefore,

$$S(R,\theta) \geq \frac{(1-\lambda)(1-br^2) - \lambda(b-1)r}{(1+r)(1+br)} .$$

Since $1 > \lambda > \frac{1}{2}$, $S(R,\theta) > 0$ for $r < \frac{-\lambda+\sqrt{\lambda^2+c(c+2-2\lambda)}}{c+2-2\lambda}$ provided

$$(6.4.13) \qquad (a+d)^2 \leq \sqrt{\frac{b(a^2-d^2)}{2\lambda-1}} .$$

One can easily check that for $r < \frac{-\lambda+\sqrt{\lambda^2+c(c+2-2\lambda)}}{c+2-2\lambda}$, (6.4.13) holds.

Suppose

$$(6.4.14) \qquad (2\lambda-1)R + \frac{b}{R} - 2a < 0.$$

Then

$$\frac{\partial S}{\partial \theta} = \frac{1}{(b-1)}\, T(R) \sin\theta$$

where $T(R) = -(2\lambda-1)R - \frac{b}{R} + 2a > 0$ (in view of 6.4.14)

Positiveness of $T(R)$ ensures that the minimum of $S(R,\theta)$ w.r.t. θ is attained at $\theta=0$

Let

$$(6.4.15) \qquad L(R) = S(R,0) = \frac{1}{(b-1)}\left[2\lambda R + \frac{b}{R} - 2a + (a^2-d^2)R^{-1} - (b+1)\lambda\right]$$

Obviously, it is a decreasing function of R for $0 \leq R \leq \sqrt{\frac{a^2-d^2+b}{2\lambda}}$. If

$$(a+d) \leq \sqrt{\frac{a^2-d^2+b}{2\lambda}}$$

then the minimum of (6.4.15) is attained at $R=a+d$ and

$$(6.4.16) \qquad L(a+d) = \frac{(1-b)(1-br^2)-r(b-1)r}{(1+br)(1+r)}$$

R.H.S. of (6.4.16) is positive for $r < \frac{-\lambda+\sqrt{\lambda^2+c(c+2-2\lambda)}}{c+2-2\lambda} = r_0$.

One can easily check that for $r = r_0$

$$(a+d) \leq \sqrt{\frac{a^2-d^2+b}{2\lambda}} .$$

Hence, $S(R,\theta) > 0$ for $r < \frac{-\lambda+\sqrt{\lambda^2+c(c+2-2\lambda)}}{c+2-2\lambda}$.

This proves that if $F(z)$ belongs to $\Sigma^*(\lambda)$, $\frac{1}{2} < \lambda < 1$, then $F(z)$ is starlike of order in $0<|z| < \frac{-\lambda+\sqrt{\lambda^2+c(c+2-2\lambda)}}{c+2-2\lambda}$.

We will show that the result is sharp for the function

$$f(z) = \frac{1}{c}\left[(c+1)F(z) + zF'(z)\right]$$

where $F(z) = \frac{(1+z)^{2(1-\lambda)}}{z}$.

If $F(z) = \frac{(1+z)^{2(1-\lambda)}}{z}$, then

$$(6.4.17) \quad -\frac{zf'(z)}{f(z)} - \lambda = (1-\lambda)\frac{1-z}{1+z} - \frac{2(1-\lambda)}{c} \cdot \frac{z}{(1+\frac{c+2-2\lambda}{c}z)(1+z)}.$$

If $0 \leq \lambda \leq \frac{1}{2}$, then one can easily see that for

$z \in \left(-\frac{c}{c+2-2\lambda}, -\frac{c}{c+2-2\lambda} - \epsilon\right)$ $(\epsilon > 0)$, the R.H.S. of (6.4.17) becomes negative.

If $\frac{1}{2} < \lambda < 1$, then for $z = \frac{-\lambda + \sqrt{\lambda^2 + c(c+2-2\lambda)}}{c+2-2\lambda}$,

$$-\text{Re}\,\frac{zf'(z)}{f(z)} - \lambda = 0.$$

Hence the result is sharp.

Corollary 6.4.1. Let $F(z)$ belongs to Σ^* and

$$f(z) = \frac{1}{c}\left[(c+1)F(z) + zF'(z)\right], \quad c > 0.$$

Then $f(z)$ is meromorphically starlike in $0 < |z| < \frac{c}{c+2}$.

Remark 6.4.1. Goel and Sohi [18] proved that if $F(z)$ belong to Σ^*, then $f(z)$ is starlike meromorphic in $0 < |z| < \sqrt{\frac{c}{c+2}}$. But their claim is incorrect. For $\frac{c}{c+2} \leq |z| < \sqrt{\frac{c}{c+2}}$, $\cos\theta$ can attain negative values and consequently $S(R,\theta)$ is not always positive. Here is a counter example to show that their claim is wrong.

Let

$$F(z) = \frac{(1+z)^2}{z}.$$

If we choose $c = \frac{2}{3}$, then according to Sohi and Goel $f(z)$ is starlike meromorphic in $0 < |z| < \frac{1}{2}$. It can be easily verified that $z = -\frac{1}{3}$

$$-\text{Re}\,\frac{zf'(z)}{f(z)} = -\frac{5}{2} < 0.$$

Hence, the claim of Sohi and Goel is incorrect.

Theorem 6.4.2. Let $F(z)$ belong to $\Sigma_K(\lambda)$ and

$$f(z) = \frac{1}{c}\,[(c+1)F(z) + zF'(z)],\ c > 0.$$

Then $f(z)$ is meromorphically convex of order λ

in $0 < |z| < \frac{c}{c+2-2\lambda}$, if $0 \leq \lambda \leq \frac{1}{2}$,

in $0 < |z| < \frac{-\lambda+\sqrt{\lambda^2+c(c+2-2\lambda)}}{(c+2-2\lambda)}$ if $\frac{1}{2} < \lambda < \frac{1}{2}$.

The result is sharp.

Corollary 6.4.2. Let $F(z)$ belong to Σ_K and

$$f(z) = \frac{1}{c}\,[(c+1)F(z) + zF'(z)],\ c > 0.$$

Then $f(z)$ is meromorphically convex in $0 < |z| < \frac{c}{c+2}$.

Theorem 6.4.3. Let $F(z)$ is close-to-convex of order and type β with respect to $G(z)$ in $0 < |z| < 1$ and

$$f(z) = \frac{1}{c}[(c+1)F(z) + zF'(z)],$$
$$g(z) = \frac{1}{c}[(c+1)G(z) + zG'(z)],\ c > 0.$$

Then $f(z)$ is close-to-convex of order λ and type β with respect to $g(z)$ in

$$0 < |z| < \frac{-(2-\beta)+\sqrt{(2-\beta)^2+c(c+2-2\lambda)}}{c+2(1-\beta)} .$$

Proof. Since $F(z)$ is close-to-convex of order λ and type β with respect to $G(z)$ in $0 < |z| < 1$, we can write

$$(6.4.18) \qquad -\frac{zF'(z)}{G(z)} = (1-\lambda)p(z)+\lambda \quad \text{where} \quad p(z) \in P$$

Differentiating (6.4.18), we get after a simple computation

$$-\frac{zf'(z)}{g(z)} - \lambda = (1-\lambda)\left[p(z)+zp'(z)\,\frac{1}{1+c+\frac{zG'(z)}{G(z)}}\right].$$

Therefore,

$$(6.4.19) \qquad -\mathrm{Re}\,\frac{zf'(z)}{g(z)} - \lambda \geq (1-\lambda)\left[\mathrm{Re}\,p(z)-|zp'(z)|\cdot\frac{1}{\left|1+c+\frac{zG'(z)}{G(z)}\right|}\right].$$

Since $G(z)$ belongs to $\Sigma^*(\beta)$, $g(z)$ is meromorphically starlike of order β

in $\quad 0 < |z| < \frac{c}{c+2-2\beta} \quad$ if $\quad 0 \leq \beta \leq \frac{1}{2}$,

in $\quad 0 < |z| < \frac{-\beta+\sqrt{\beta^2+c(c+2-2\beta)}}{c+2-2\beta} \quad$ if $\quad \frac{1}{2} < \beta < 1$.

One can easily verify that

$$\frac{-(2-\beta)+\sqrt{(2-\beta)^2+c(c+2-2\beta)}}{c+2(1-\beta)} \leq \frac{c}{c+2-2\beta}$$

also,

$$\frac{-(2-\beta)+\sqrt{(2-\beta)^2+c(c+2-2\beta)}}{c+2(1-\beta)} \leq \frac{-\beta+\sqrt{\beta^2+c(c+2-2\beta)}}{c+2-2\beta} .$$

Therefore, it is sufficient to show that

$$-\mathrm{Re}\, \frac{zf'(z)}{g(z)} > \lambda \text{ for } |z| < \frac{-(2-\beta)+\sqrt{(2-\beta)^2+c(c+2-2\beta)}}{c+2-2\beta} .$$

Since $G(z)$ belongs to $\Sigma^*(\beta)$, there exists a $p_1(z) \in P$ such that

$$-\frac{zG'(z)}{G(z)} = (1-\beta)p_1(z) + \beta$$

$p_1(z) \in P$ therefore,

$$(6.4.20) \qquad \left|-p_1(z) + \frac{1+r^2}{1-r^2}\right| \leq \frac{2r}{1-r^2} \quad (|z|=r).$$

Let

$$h(z) = \frac{1}{(1+c)-((1-\beta)p_1(z)+\beta)} .$$

Then

$$\left|\frac{1}{h(z)} - ((1+c-\beta) - a(1-\beta))\right| < d(1-\beta)$$

where $a = \frac{1+r^2}{1-r^2}$ and $d = \frac{2r}{1-r^2}$,

i.e. $\left|\frac{1}{h(z)} - S\right| \leq T$

where $S = [(1+c-\beta)-a(1-\beta)]$ and $T = d(1-\beta)$.

If $r < \frac{c}{c+2(1-\beta)}$, then $|S|-T > 0$.

Hence,

$$\frac{1}{|S|-T} \geq |h(z)| \geq \frac{1}{|S|+T}.$$

Application of (6.4.20) in (6.4.19) provides

$$(6.4.21) \qquad \text{-Re}\,\frac{zf'(z)}{g(z)} - \lambda \geq (1-\lambda)\left[1 - \frac{T}{(1-\beta)}\,\frac{1}{|S|-T}\right] \text{Re}\; p(z)$$

(3.29) provides that

$$\text{-Re}\,\frac{zf'(z)}{g(z)} - \lambda > 0, \text{ for } |z| < \frac{-(2-\beta)+\sqrt{(2-\beta)^2+c(c+2-2\beta)}}{c+2-2\beta}.$$

This completes the proof the theorem.

<u>Corollary 6.4.3.</u> Let $F(z)$ be close-to-convex of order λ with respect to $G(z)$ in $0 < |z| < 1$ and

$$f(z) = \frac{1}{c}[(c+1)F(z) + zF'(z)],\; g(z) = \frac{1}{c}[(c+1)G(z) + zG'(z)]$$

If $c > 0$, then $f(z)$ is close-to-convex of order λ with respect to $G(z)$

in $0 < |z| < \frac{-2+\sqrt{c^2+2c+4}}{c+2}$.

<u>Remark 6.4.2.</u> The result of Sohi and Goel [18] follow from Theorem 6.4.3 by taking $\lambda = 0$ and $\beta = 0$.

We could not claim the sharpness of the result in Theorem 6.4.3 . However, this theorem can be restated without reference to the functions $g(z)$ and $G(z)$.

<u>Theorem 6.4.4.</u> Let $F(z)$ be regular in $0 < |z| < 1$ and

$$\text{-Re}\; z^2F'(z) > \lambda \qquad \text{for } |z| < 1.$$

If

$$f(z) = \frac{1}{c}[(c+1)F(z)+zF'(z)],\; c > 0,$$

then

$$-\operatorname{Re} z^2 f'(z) > \lambda \quad , \text{ for } 0 < |z| < \frac{-1+\sqrt{1+c^2}}{c} .$$

The result is sharp.

<u>Proof</u>. Let

$$-z^2 F'(z) = (1-\lambda)p(z) + \lambda \qquad \text{where} \quad p(z) \in P .$$

Then, we have

$$-z^2 f'(z) = \frac{1}{c}\left[(c+2)z^2F'(z) + z^3F''(z)\right]$$

$$-z^2 f'(z) - \lambda \geq (1-\lambda)p(z) + \frac{1-\lambda}{c} z p'(z)$$

$$-\operatorname{Re} z^2 f'(z) - \lambda \geq \frac{1-\lambda}{c}\left[c \operatorname{Re} p(z) - |zp'(z)|\right]$$

Using (6.4.20), we obtain

$$-\operatorname{Re} z^2 f'(z) - \lambda \geq \frac{1-\lambda}{c}\left[c - \frac{2r}{1-r^2}\right] \operatorname{Re} p(z) > 0,$$

$$\text{for} \quad 0 < |z| < \frac{\sqrt{1+c^2}-1}{c}$$

The result is sharp for the function

$$f(z) = \frac{1}{c}\left[(c+1)F(z)+zF'(z)\right] \quad , \ c > 0$$

where $-z^2F'(z) = \dfrac{1+(1-2\lambda)z}{1-z}$.

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